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![A graph](image)

**Figure:** A graph

There is a more formal definition which gives the vertices of a graph, and then specifies which vertices have a line between them (in a mathy way).
Figure: The Petersen graph
Figure: The Petersen graph with vertices labeled
Figure: Octahedron  

Figure: Icosahedron  

Figure: Dodecahedron  

When are two graphs really the same?
The definition of a graph does not specify how a graph should be drawn, or give any information about what a line looks like.

Figure: Two drawings of the same graph
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![Two drawings of the same graph](image)

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So we can draw an edge in any shape, length or color, and that does not change the graph.
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Formally, we say that two graphs that are really the same are **isomorphic**. The roots of this word are Greek, where iso means “same”, and morph means “form or shape”.
When trying to see if two graphs are really the same, usually the first thing one looks for are differences (if the graphs are really different, or not isomorphic) or similarities (if the graphs are really the same, or isomorphic) in “structural properties”.

Well, it is something that doesn’t change no matter how the graph is drawn or labeled. That sounds a lot like asking if the two graphs are the same. But a structural property usually refers not to the whole graph, but some piece of information about the graph or even small parts of the graph itself. We will now give some examples. We start with the most basic, which hopefully will be obvious to you, and then look at more complicated things.
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The easiest way to see that two graphs are NOT isomorphic, is if they have different numbers of vertices or edges.
Number of edges and vertices

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How many pieces is the graph made out of?

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![Graphs and figures]

**Figure:** A graph that is two pieces

**Figure:** A graph that is one piece
How many pieces is the graph made out of?

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![Figure: A graph that is two pieces](image1)

![Figure: A graph that is one piece](image2)

Notice that the number of vertices of these two graphs are different. These graphs differ in other structural properties, and so there are other reasons why these graphs are not isomorphic.
A cycle in a graph is a sequence of vertices, say, $v_1, v_2, \ldots, v_r, v_1$ which, except for the first and last vertices, are all different, and there is an edge in the graph between successive pairs (i.e. there is an edge between $v_1$ and $v_2$, $v_2$ and $v_3$, etc., and $v_r$ and $v_1$). Such a cycle is usually called an $r$-cycle. Here are some examples:
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![Figure: A 5-cycle or pentagon](image2)

![Figure: A 6-cycle or a hexagon](image3)
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Another definition

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Are the valencies the same?

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Moving a vertex to a different position in a graph or relabeling it doesn’t change its valency. We usually describe the valencies of all of the vertices in a graph with a list of all valencies, typically from smallest valency to largest valency.
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Moving a vertex to a different position in a graph or relabeling it doesn’t change its valency. We usually describe the valencies of all of the vertices in a graph with a list of all valencies, typically from smallest valency to largest valency. If the lists are the same, the graphs MIGHT be isomorphic. If the lists are not the same, they are not isomorphic.

These graphs are not isomorphic as on the right hand graph every vertex of degree 3 forms a cycle while on the left hand side they do not.

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When are two graphs really the same?

Figure: A graph with degree sequence 
(2, 2, 2, 2, 3, 3, 3, 3)

Figure: Another graph with degree sequence 
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Moving a vertex to a different position in a graph or relabeling it doesn’t change its valency. We usually describe the valencies of all of the vertices in a graph with a list of all valencies, typically from smallest valency to largest valency. If the lists are the same, the graphs MIGHT be isomorphic. If the lists are not the same, they are not isomorphic.

These graphs are not isomorphic as on the right hand graph every vertex of degree 3 forms a cycle while on the left hand side they do not.
If none of the previous techniques we have discussed give that a pair of graphs are not isomorphic, it is probably time to focus on some small graph that is part of both the graphs and see if this can help. It is usually a good idea to choose something that is not too small (there could be too many of them to do much good) and not too large (they can be hard to find).
If none of the previous techniques we have discussed give that a pair of graphs are not isomorphic, it is probably time to focus on some small graph that is part of both the graphs and see if this can help. It is usually a good idea to choose something that is not too small (there could be too many of them to do much good) and not too large (they can be hard to find). Cycles are a good feature to focus on.
Are these graphs isomorphic?
Are these graphs isomorphic?
Yes, they are!
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When are two graphs really the same?
Are these two graphs isomorphic?
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Where is the hard part?

If you think back to the examples we have seen, to show two graphs are NOT isomorphic, we look for distinguishing features of the graphs, and then compare them.
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What is the state of the art?

In 2015, László Babai used the framework developed by Luks to show that the graph isomorphism problem could be solved not quickly, but not really slowly (technically, he showed the problem could be solved in quasipolynomial time).
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Thanks!