Famnitovi izleti v matematično vesolje: "To Infinity and Beyond"

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Motivation

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Infinite, infinite, infinite, infinite, . . .

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Infinite, infinite, infinite, infinite, . . .

Source: NASA

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Infinite, infinite, infinite, infinite, . . .

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Is the universe infinite?

- Aristotle (350 BC) distinguished **potential infinity** (adding $1+1+1+1+\cdots$) from **actual infinity** (real entities) for which he postulated to be impossible to exist.

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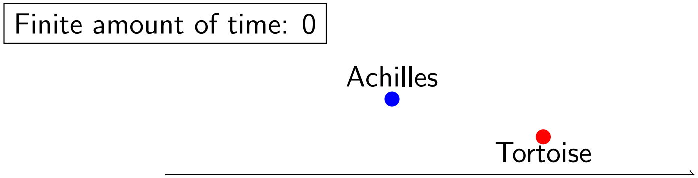
- The Hellenistic Greeks were terrified of the infinite. However, Euclid proved that **there are infinitely many prime numbers** (avoiding the word infinity).

Paradoxes I

(Arguably) The most famous paradox about infinity: Zeno's Paradox.

"The Race Between Achilles and the Tortoise"



In mathematical terms, if a is Achilles's speed in meters per second and x the tortoise's speed ($a > x$), then Achilles needs $\frac{100}{a}$ seconds to reach the tortosie initial position, In mathematical terms, if a is Achilles's speed in meters per second and x the tortoise's speed ($a > x$), then Achilles needs $\frac{100}{a}$ seconds to reach the tortosie initial position, $\frac{100}{a^2}x$ seconds to reach the tortoise's second position, In mathematical terms, if a is Achilles's speed in meters per second and x the tortoise's speed ($a > x$), then Achilles needs $\frac{100}{a}$ seconds to reach the tortosie initial position, $\frac{100}{a^2}x$ seconds to reach the tortoise's second position, then $\frac{100}{a^3}x^2$, and so on.

In mathematical terms, if a is Achilles's speed in meters per second and x the tortoise's speed $(a > x)$, then Achilles needs $\frac{100}{a}$ seconds to reach the tortosie initial position, $\frac{100}{a^2}x$ seconds to reach the tortoise's second position, then $\frac{100}{a^3}x^2$, and so on. In total, he needs

$$\frac{100}{a}\sum_{n=0}^{\infty}(\frac{x}{a})^n$$

seconds to reach the tortoise.

We don't have to go this far: **try to express $\frac{1}{3}$ as a decimal.**

17th - 19th Centuries

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- With the invention of Calculus by Isaac Newton and Gottfried Leibniz—infinitesimals, very small quantities.
- In 1655, John Wallis introduced the lemniscate symbol ∞ to compute areas.
- The great mathematician Leonhard Euler devised the importance of the infinite and provided theorems about infinite sums and products... **Without a proper definition of either convergency or infinity!**

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Intuitively, the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

is infinite. How to define it?

In modern times

In modern times

In modern times

In modern times

In modern times

In modern times

In modern times

Aleksandra

In modern times

Aleksandra Blaž

In modern times

Aleksandra Blaž Marija

In modern times

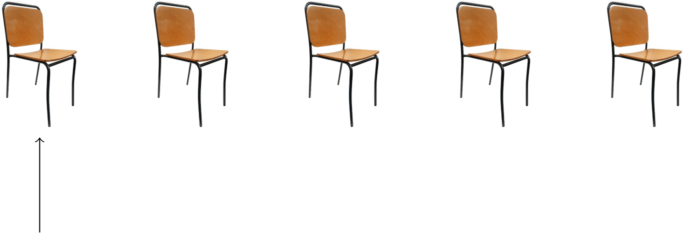
Aleksandra Blaž Marija Nika

In modern times

Aleksandra Blaž Marija Nika Žiga

In modern times



Aleksandra Blaž Marija Nika Žiga

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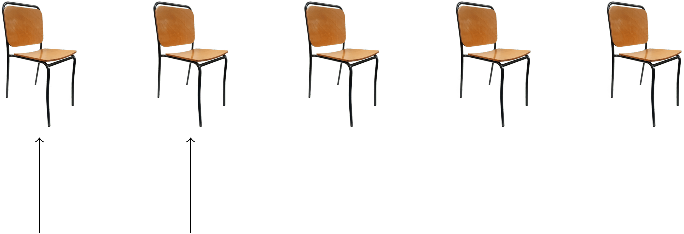

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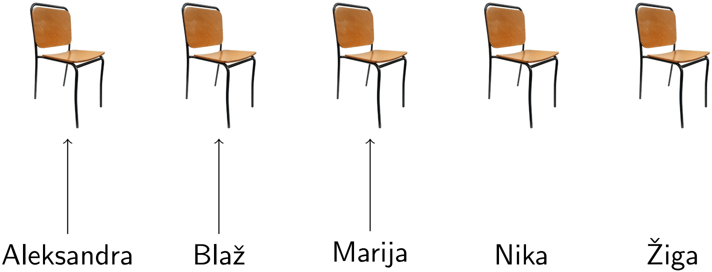

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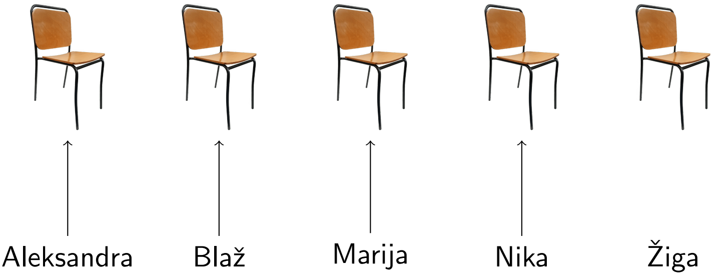

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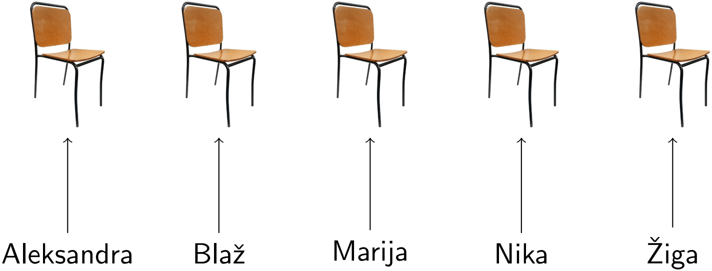

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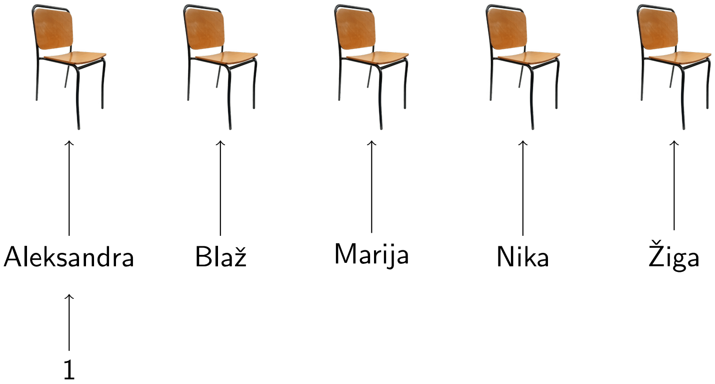

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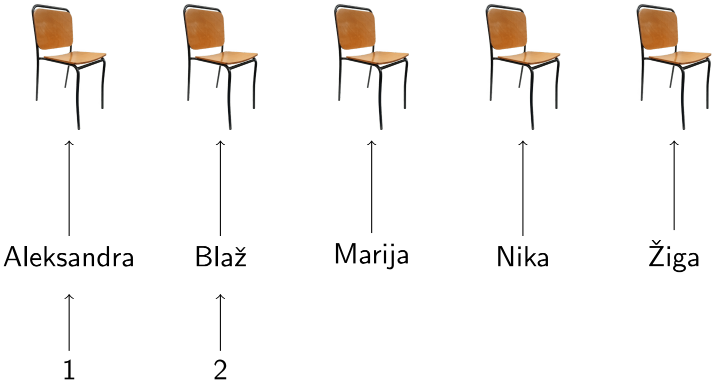

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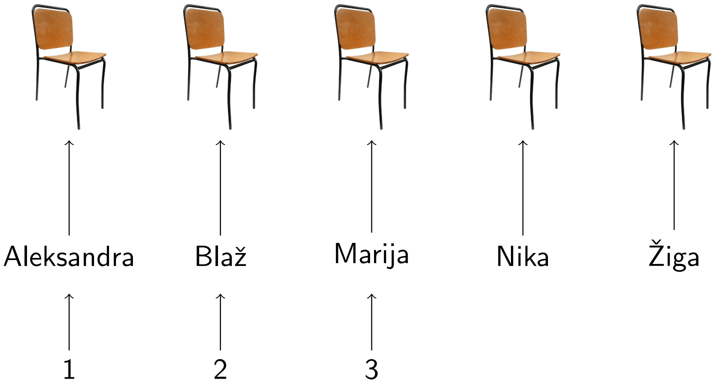

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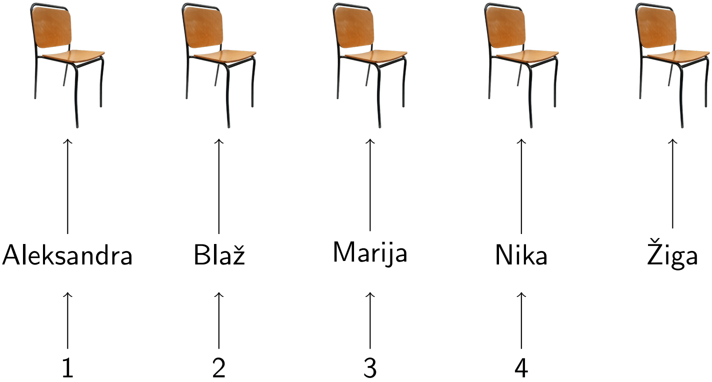

In modern times

↑	↑	↑	↑	↑
Aleksandra	Blaž	Marija	Nika	Žiga
↑	↑	↑	↑	↑
1	2	3	4	5

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Figure: **Georg** Ferdinand Ludwig Philipp **Cantor**
Source: University of Hamburg

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A set X is finite if there exists a natural number $n \in \mathbb{N}$ such that there is a bijection between X and $\{0, 1, \dots, n\}$. If such bijection doesn't exist, we say that X is infinite.

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Examples: $\mathbb{N}$, the set of real numbers $\mathbb{R}$, the set of complex numbers $\mathbb{C}$.

Two sets X and Y have the same size, denoted by $|X| = |Y|$ if and only if there's a bijection between them. If there is an *injective* function and $|X| \neq |Y|$, we denote it by $|X| < |Y|$.

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For instance, $(1,0,1,1,1,1,1,1,\ldots) \in \{0,1\}^{\mathbb{N}}$.

The set $\{0,1\}^{\mathbb{N}}$ contains a "copy" of $\mathbb{N}$:

$$(1,0,0,0,0\ldots),(1,1,0,0,0\ldots),(1,1,1,0,0\ldots),\ldots$$

so it is infinite.

Moreover, there is no bijection from $\mathbb{N}$ onto $\{0,1\}^{\mathbb{N}}$ because **if there were a bijective function**

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Define the infinite binary sequence a_∞ that differs at the i-th bit from a_i. This a_∞ is not covered by any of the a_i's in the list.
Thus, such enumeration is impossible!

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By applying a similar argument, **we can create an infinite list of infinities:**

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} \cdots$$

$\aleph$'s and Cardinal Arithmetic

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Similarly, for multiplication...

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Cantor's problem: Is it true that for any infinite subset of real numbers A, either $A = |\mathbb{N}|$ or $A = |\mathbb{R}|$?

?

?

Shoes and Socks

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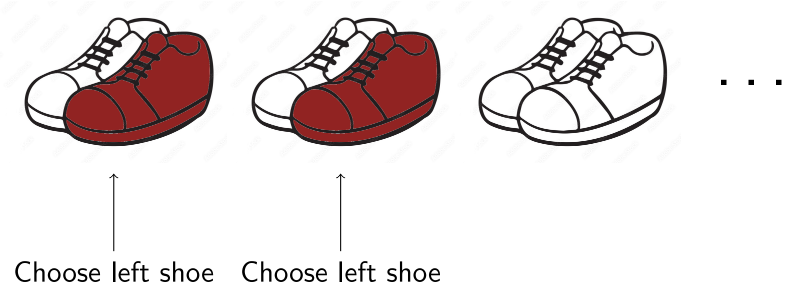
. . .

Shoes and Socks

. . .

Choose left shoe

Shoes and Socks



Shoes and Socks

   . . .

Choose left shoe Choose left shoe Choose left shoe

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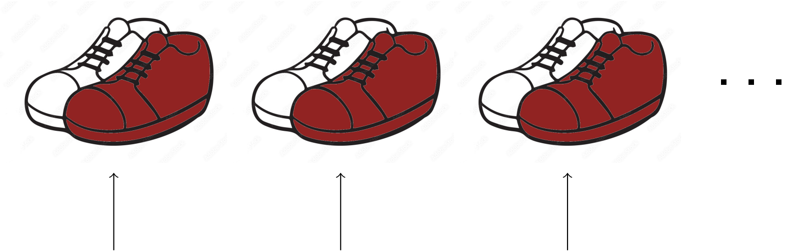


Choose left shoe   Choose left shoe   Choose left shoe

This is a well-determined choice function.

What about socks?

What about socks?

What about socks?

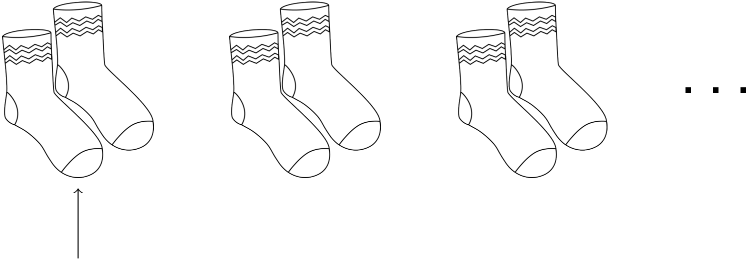
What about socks?

What about socks?

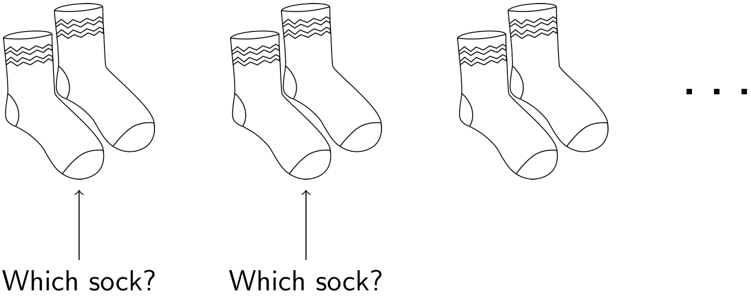
. . .

What about socks?



Which sock?

What about socks?



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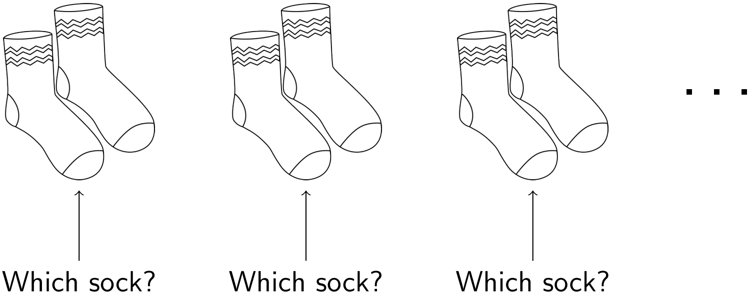

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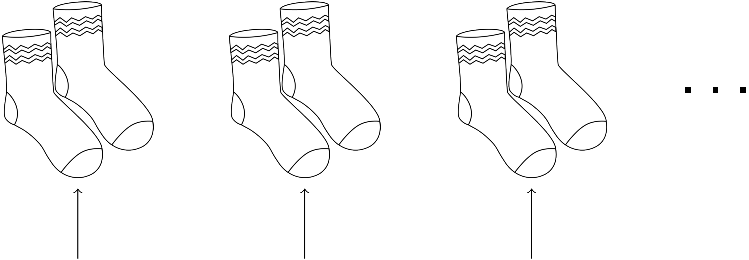

Which sock? Which sock? Which sock?

In this case, there's not an easy way to create a choice function.

Zermelo's Axiom of Choice

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In the beginning, the axiom of choice was controversial. Nowadays, it's freely used in mathematics.

The Banach-Tarski Paradox

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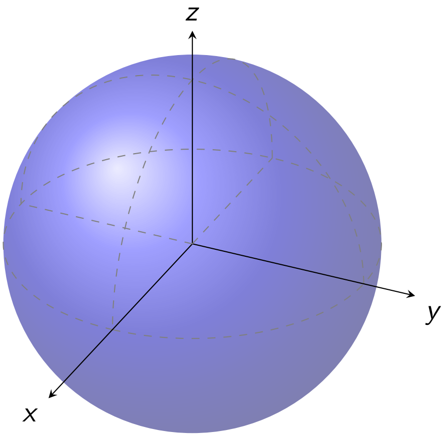

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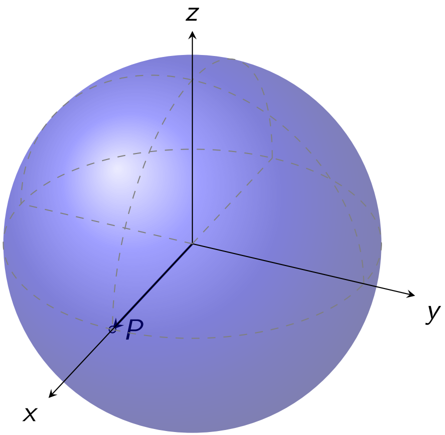

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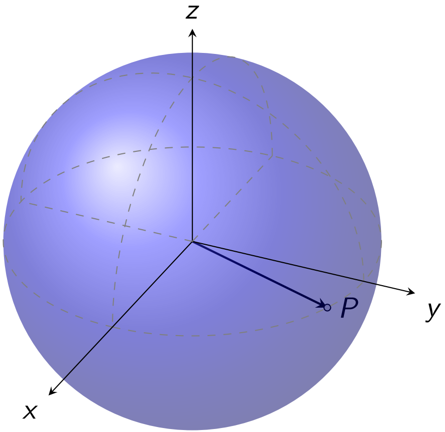
Rotation B: The point $P = (1, 0, 0)$ goes to $(\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0)$



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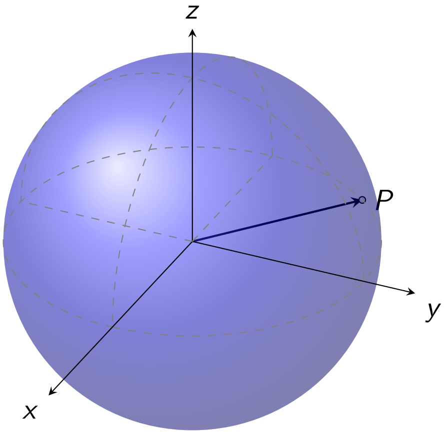
Rotation B^2: The point $P = (1, 0, 0)$ goes to $(\frac{-7}{9}, \frac{4\sqrt{2}}{9}, 0)$



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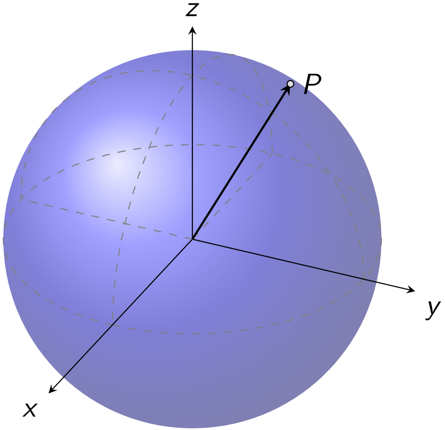
Rotation AB^2: The point $P = (1, 0, 0)$ goes to $(\frac{-7}{9}, \frac{4\sqrt{2}}{27}, \frac{16}{27})$



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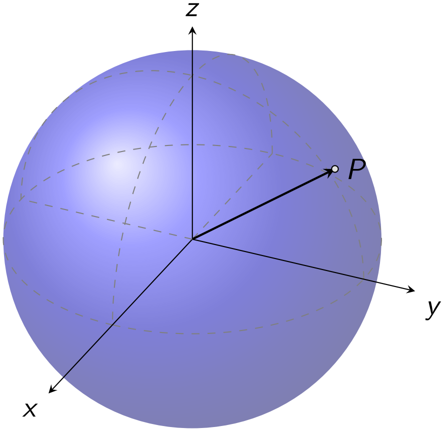
Rotation $B^{-1}AB^2$: The point $P = (1, 0, 0)$ goes to $(\frac{-5}{81}, \frac{46\sqrt{2}}{81}, \frac{16}{27})$.



We can see that the point $(1, 0, 0)$ is never reached back by these combinations of rotations! Therefore, these are essentially all "words" in the letters A, B, A^{-1}, B^{-1} (by canceling out suitable terms like AA^{-1}):

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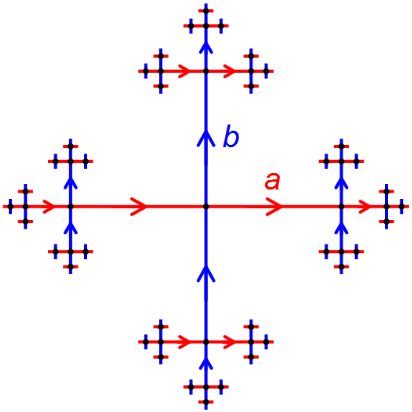

Source: Wikipedia

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Collect such words (rotations) and form the set G.

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Starting from every point $\mathbf{x} = (x, y, z)$ in the sphere, collect all points that can be reached by $\mathbf{x}$ in $\mathcal{O}(\mathbf{x})$, i.e.

$$\mathcal{O}(\mathbf{x}) := \{g\mathbf{x} | g \in G\}.$$

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$$\mathcal{O}(\mathbf{x}) := \{g\mathbf{x} | g \in G\}.$$

Consider *all* sets of the form $\mathcal{O}(\mathbf{x})$. Then each of these represents a piece of the sphere. So we have cut the sphere in a bunch of pieces!

Crucial step: Choose one element from each of the elements in the previous partition and collect them in a set, say, C.

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Using C, we can create another partition of S^2 with sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ in such a way that:

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$$B\mathcal{P}_2 = \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4, \; A\mathcal{P}_3 = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$$

This is just another partition of the sphere!


Adobe Stock | #33259157

This means that after rotating $\mathcal{P}_2$ by B and $\mathcal{P}_3$ by A and reassembling together with $\mathcal{P}_1$ and $\mathcal{P}_4$, we get two copies of the initial sphere!

Wait... What?

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Applying the same procedure (finitely) many times, one can essentially partition a pea and turn it into the sun!

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"At the end of the (chosen) day, it's your choice to choose choice or not to choose choice."

Paradoxes II

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Questions like: How to formally construct the set $\mathbb{N}$?

More importantly, what on earth is a set?

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Well, then no. If R does not belong to R, then by definition of R, $R \in R$. Again, a contradiction.

Conclusion: R cannot be a set!

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Different philosophical schools of thought rose, among them:

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Figure: Luitzen Egbertus Jan "Bertus" Brouwer, Source: St. Andrews University

Axiomatization of Mathematics—Hilbert's Program

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In such a way that mathematics are complete (all truths can be proved) and consistent (no contradictions).

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Why? There are two forms to negate this axiom.

Hyperbolic Geometry

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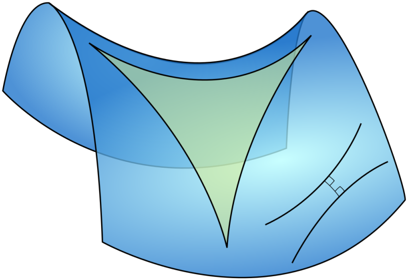

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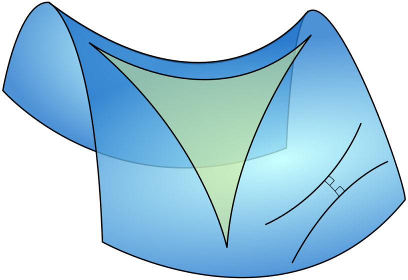

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Lobachevsky first proposed and study its properties.

Elliptic Geometry

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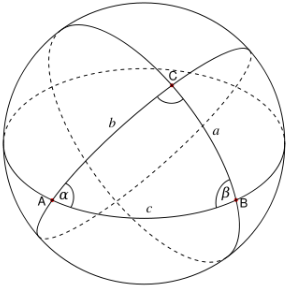

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Is this statement true?

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Is this statement true? It depends on the model (=where we interpret the symbols).

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Figure: **David Hilbert**
Source: St. Andrew University

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No problem, we can live with this...

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Figure: Kurt Gödel
Source: University of Bonn

Therefore, Hilbert's program is doomed to fail... **In an axiomatic framework, we have to live with the fact that there are undecidable things and that we can't prove the consistency of our theory.**

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Moreover, we have **relative consistency proofs**: Assuming the consistency of ZF, we can prove if a statements is consistent with it.

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Recall the axiom of choice (AC) and our initial problem (Cantor's problem or CH): **Is it true that for any infinite subset of real numbers A, either $A = |\mathbb{N}|$ or $A = |\mathbb{R}|$?**

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Gödel (1938): The axiom of choice is consistent with ZF.

Gödel (1940): CH is consistent with ZF.

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So we cannot prove or disprove these two statements!!!

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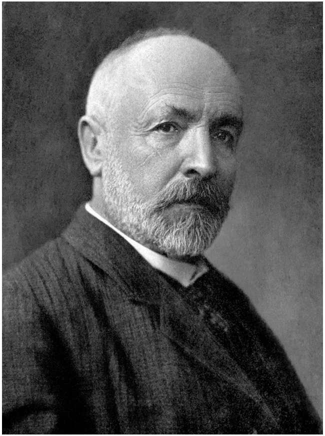

Figure: Older Cantor
Source: Carnegie Mellon University

“The essence of mathematics lies precisely in its freedom.”–Cantor

"The essence of mathematics lies precisely in its freedom."–Cantor

Thanks!