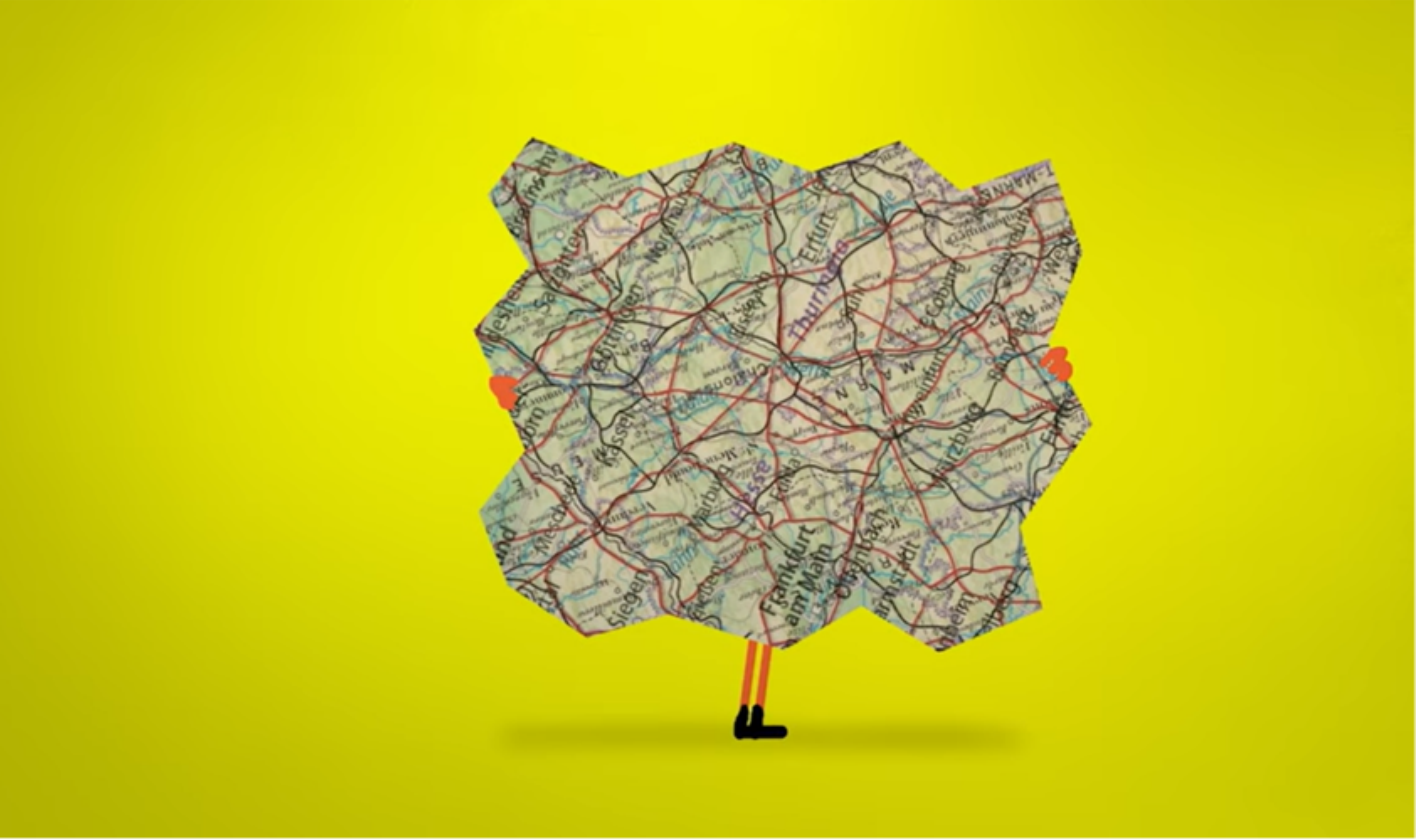
Königsberg:

A Hidden Key to Graph Theory's Door

February 21, 2024, Koper, Slovenia

FAMNITovi izleti v matematično vesolje

Blas Fernández
blas.fernandez@famnit.upr.si



CLOSE YOUR EYES AND THINK ABOUT A REALLY BIG MAP SHOWING EVERY CITY ON EARTH!

YOU WOULD HAVE A HARD TIME FINDING A PLACE CALLED KÖNIGSBERG ...

BUT ONE PARTICULAR CHARACTERISTIC OF ITS GEOGRAPHY, HAS MADE IT ONE OF THE MOST FAMOUS CITIES IN MATHEMATICS!

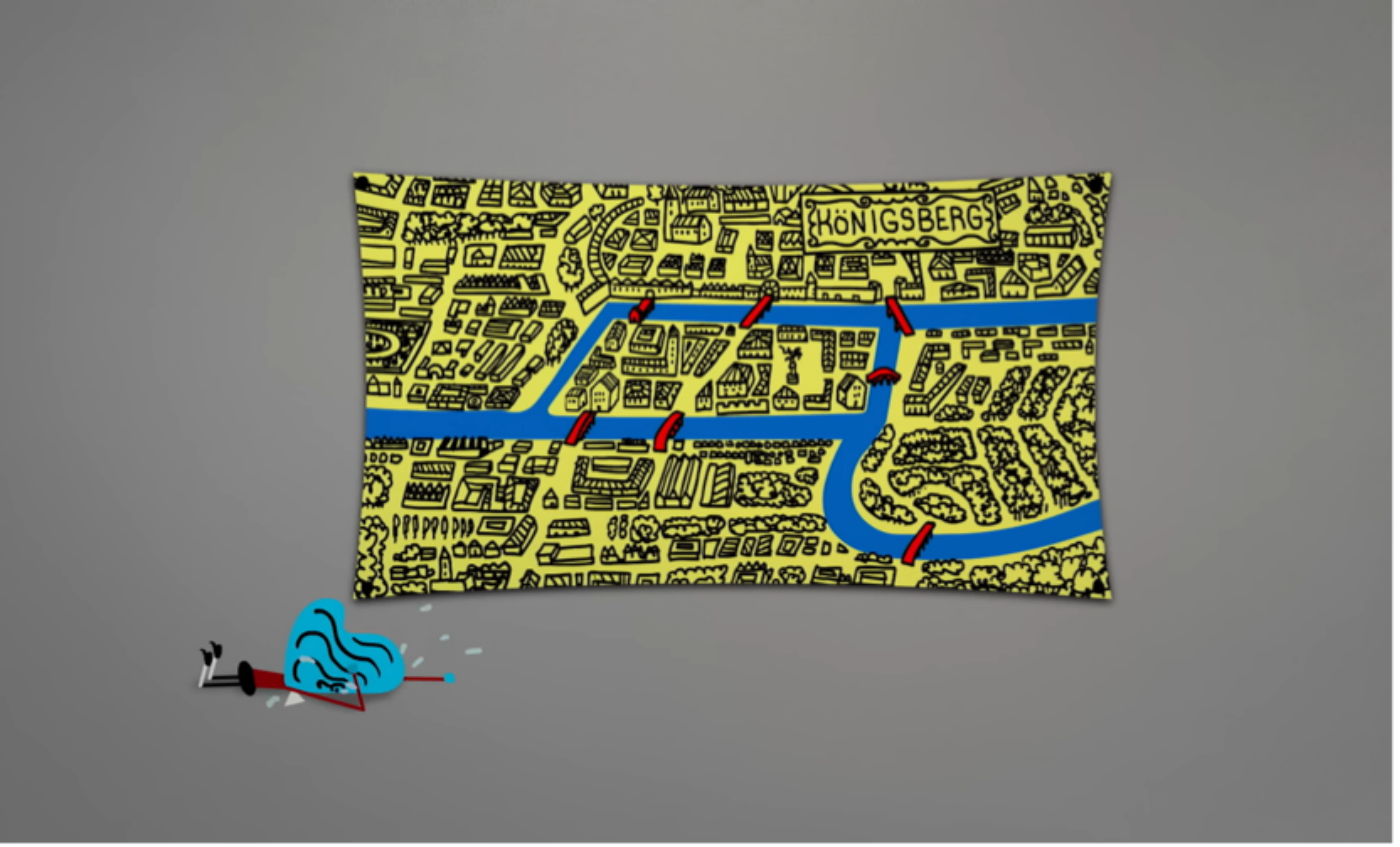
IN THE OLD GERMAN CITY OF KÖNIGSBERG WE HAVE A RIVER CALLED PREGEL.

THE CITY SITS ON BOTH SIDES OF THAT RIVER.

IN THE MIDDLE OF THE CITY, THERE ARE TWO BIG ISLANDS.

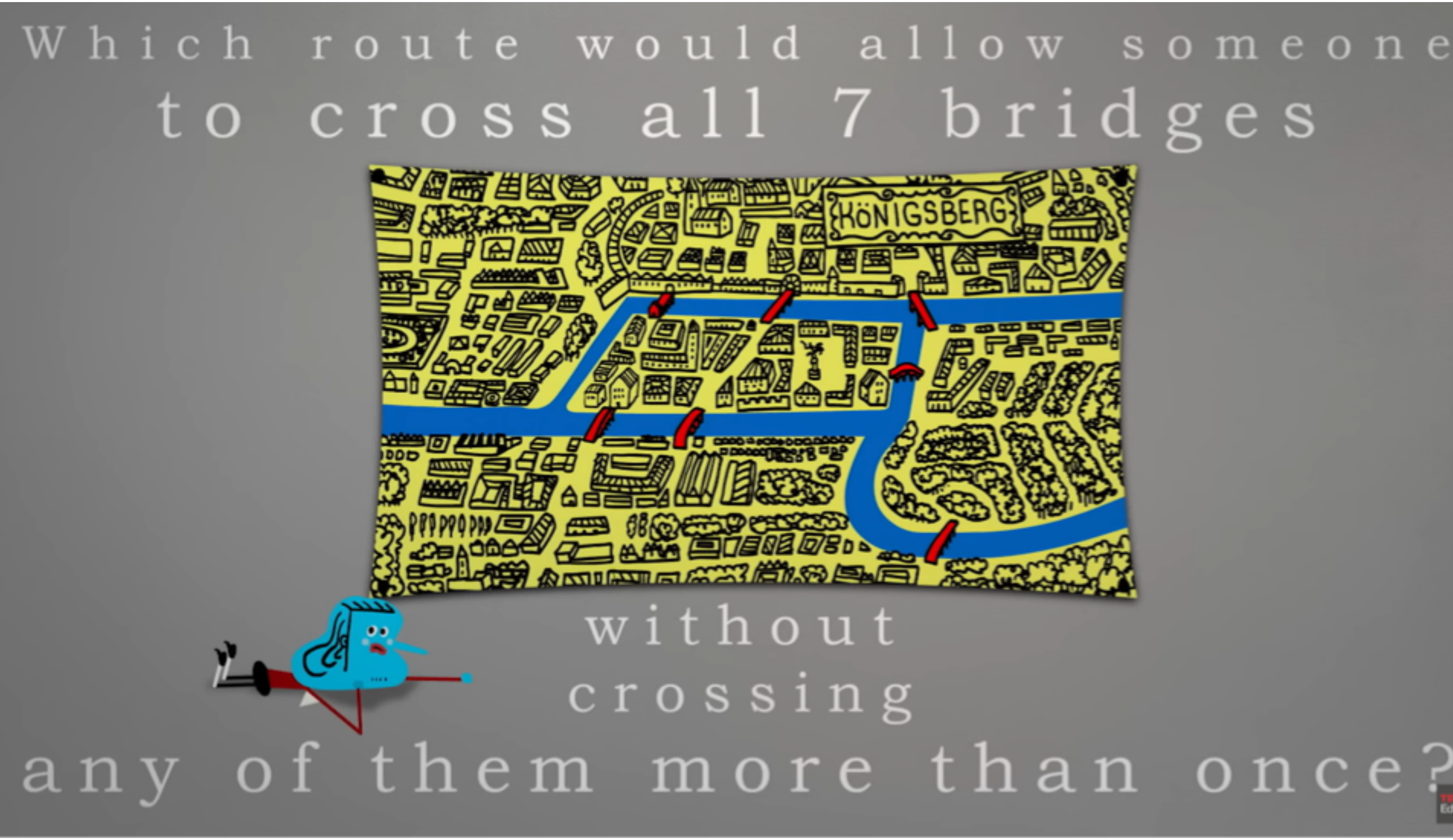
THESE ISLANDS ARE LINKED TOGETHER AND TO THE RIVERBANKS BY SEVEN BRIDGES...

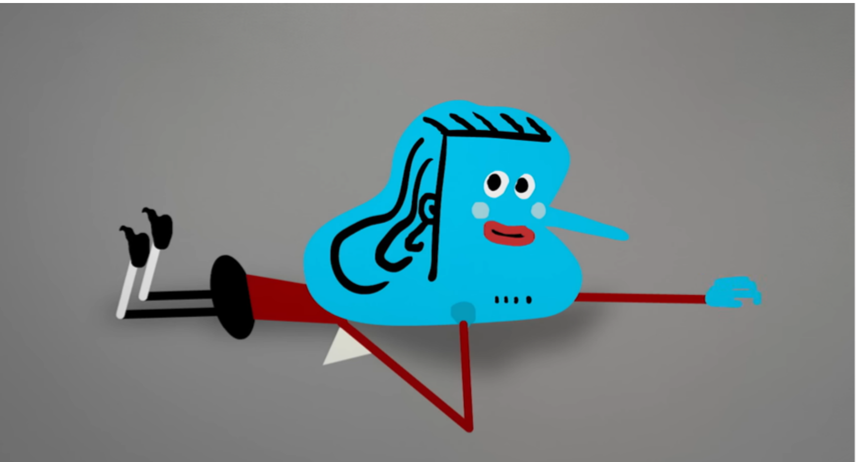
A MATHEMATICIAN WHO LATER BECAME THE MAYOR OF A NEARBY TOWN ...



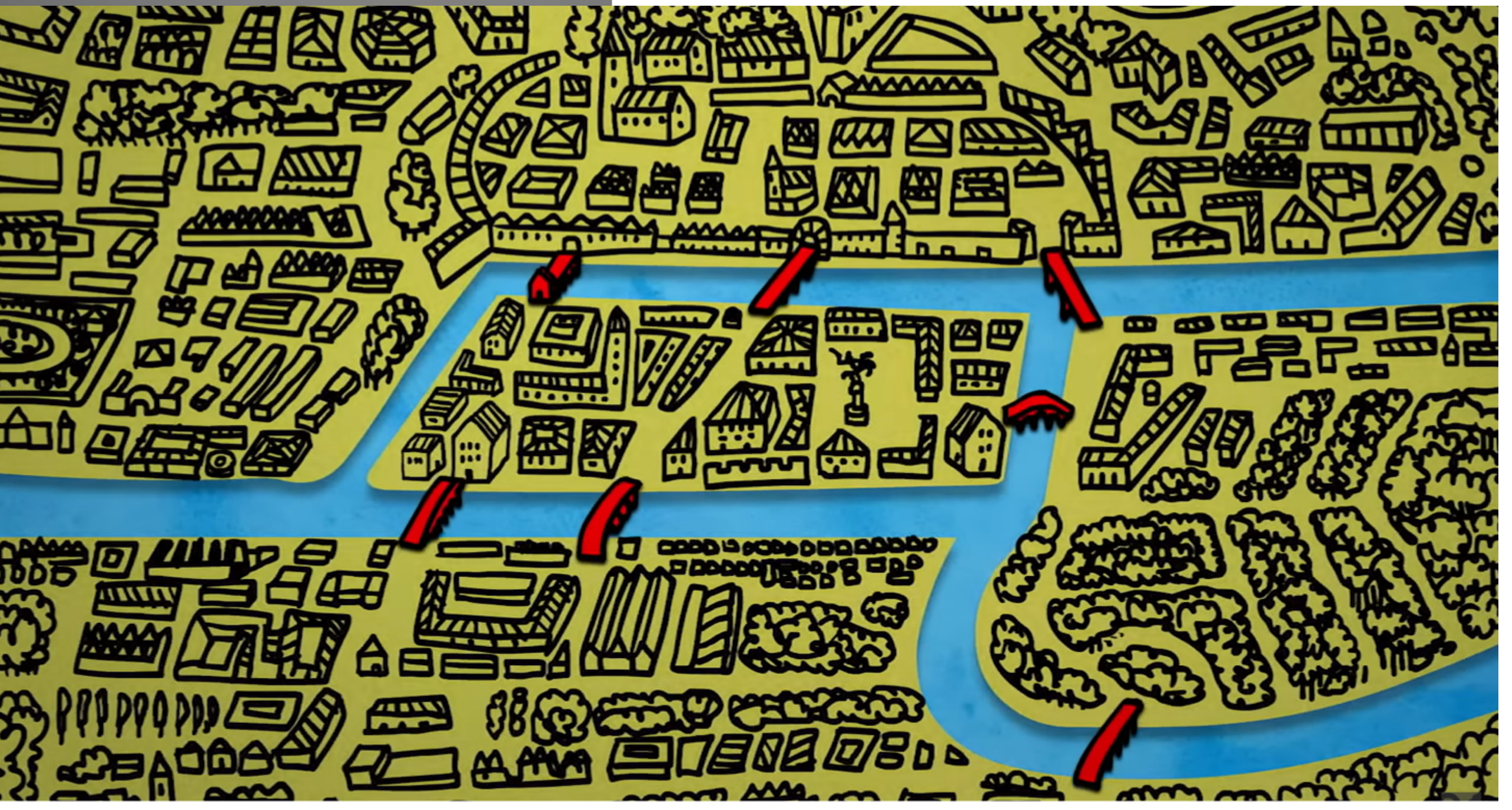
... GREW OBSESSED WITH

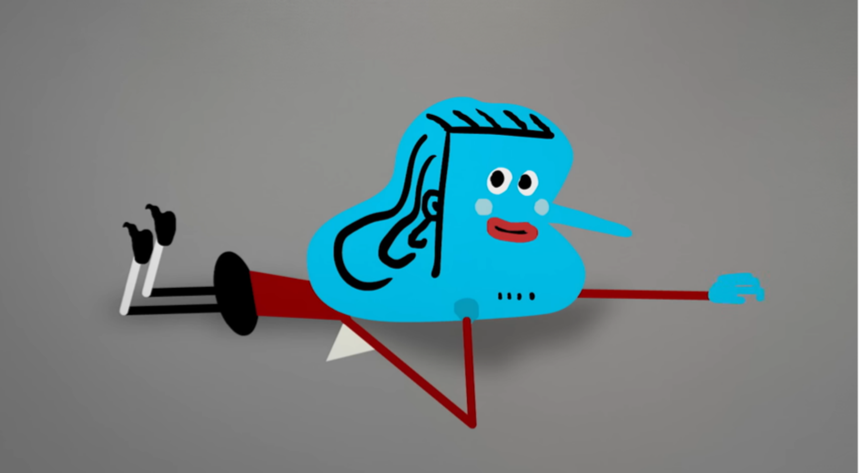
THESE ISLANDS AND BRIDGES ...


Which route would allow someone
to cross all 7 bridges
KÖNIGSBERG
without
crossing
any of them more than once?

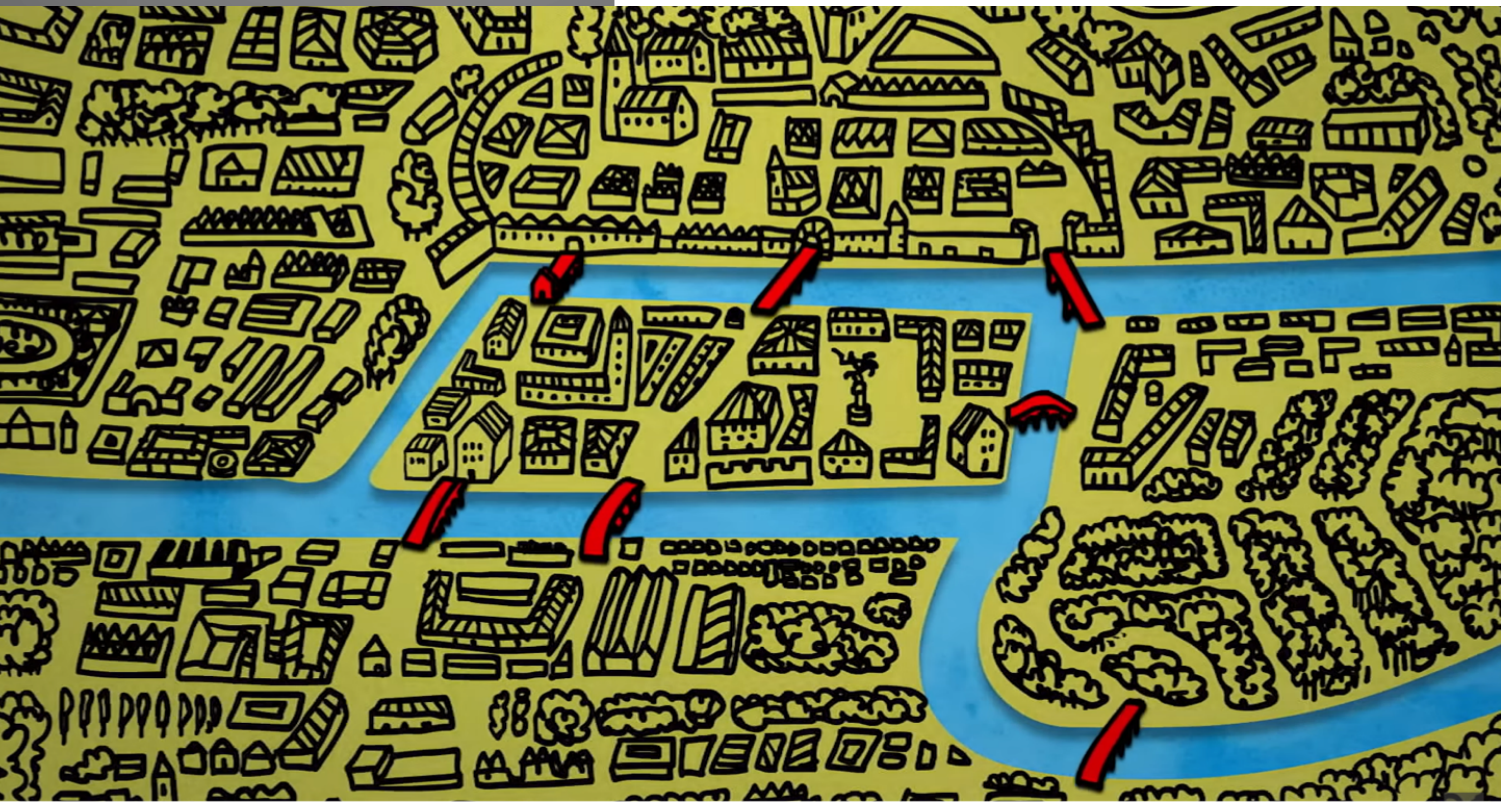


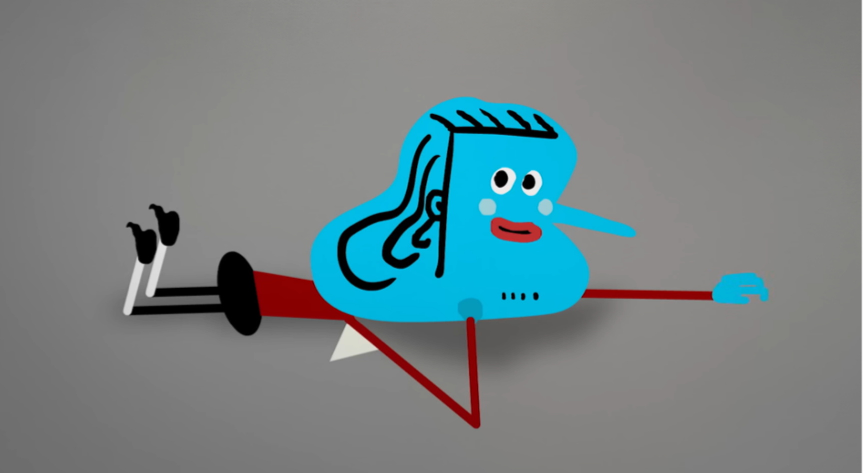
THINK ABOUT IT!





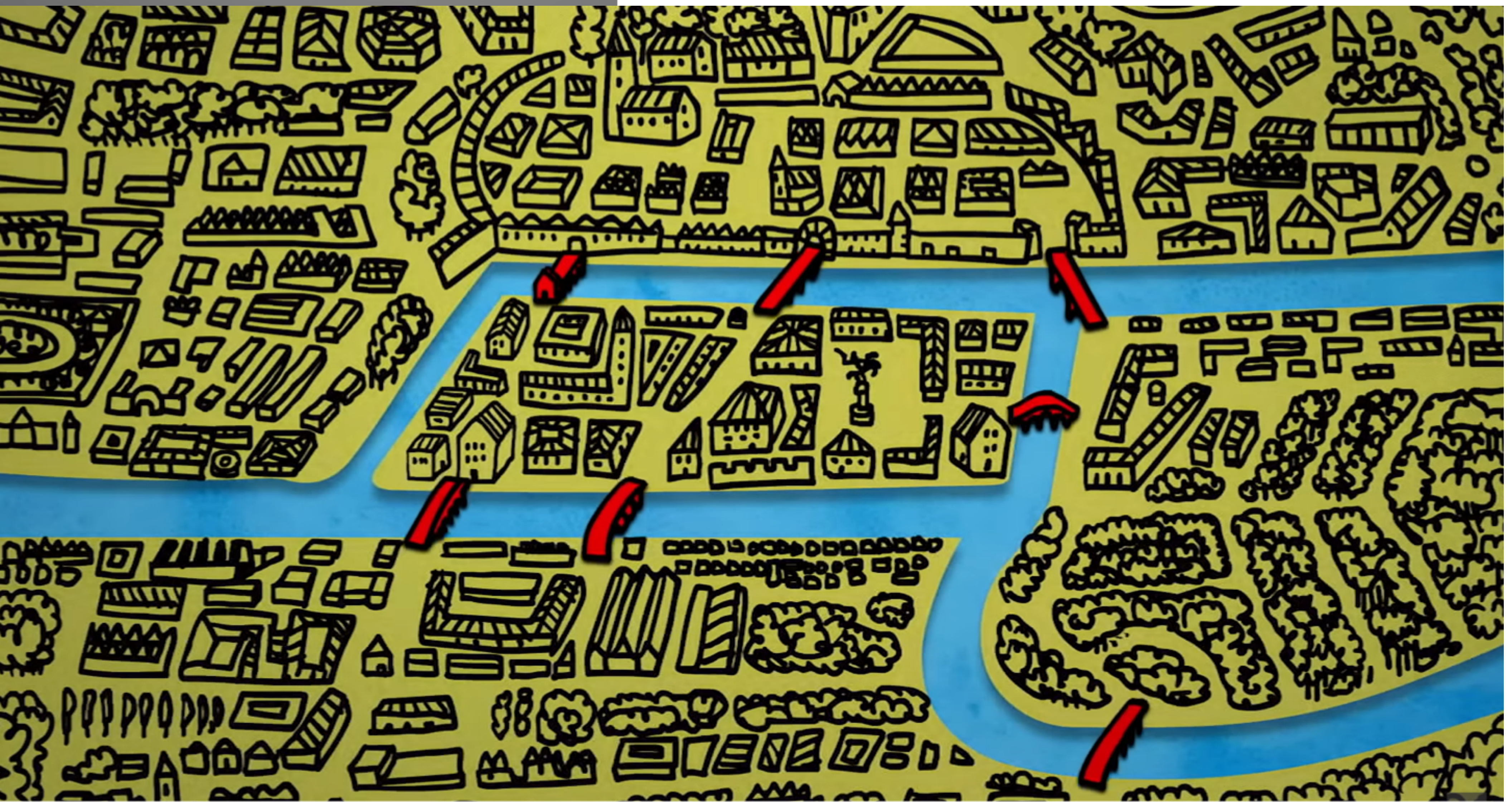
You Give up?

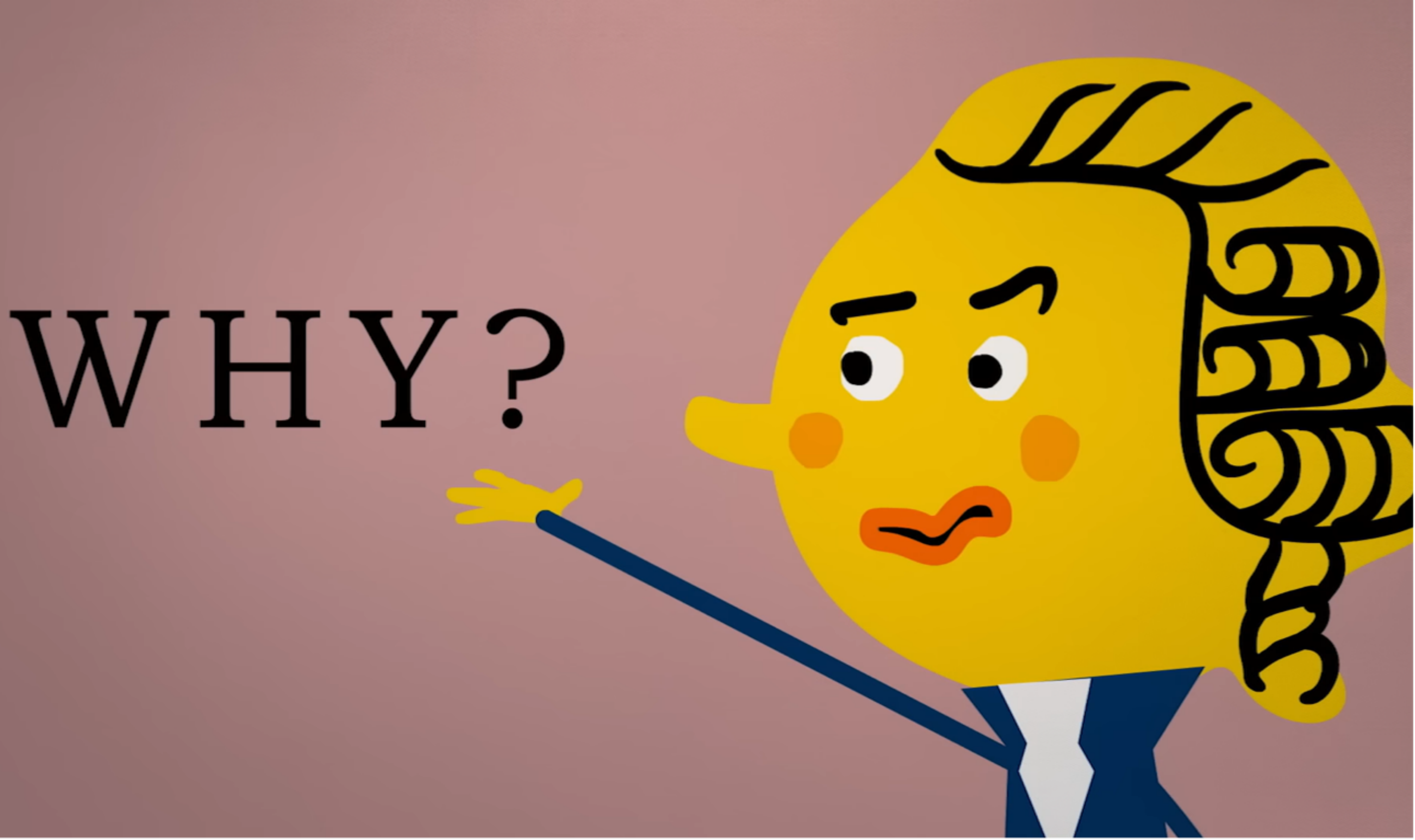




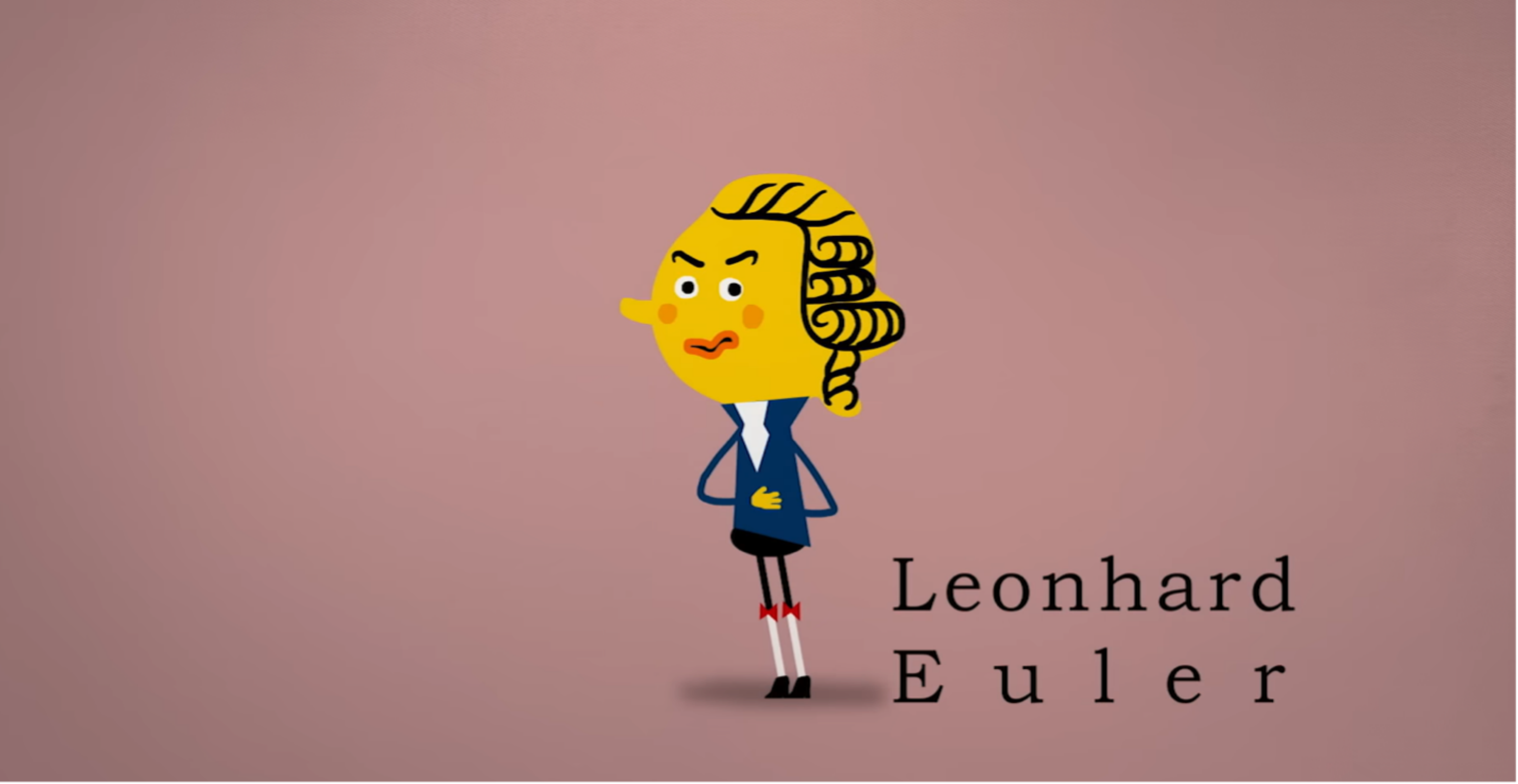
You Should...

It's not Possible!

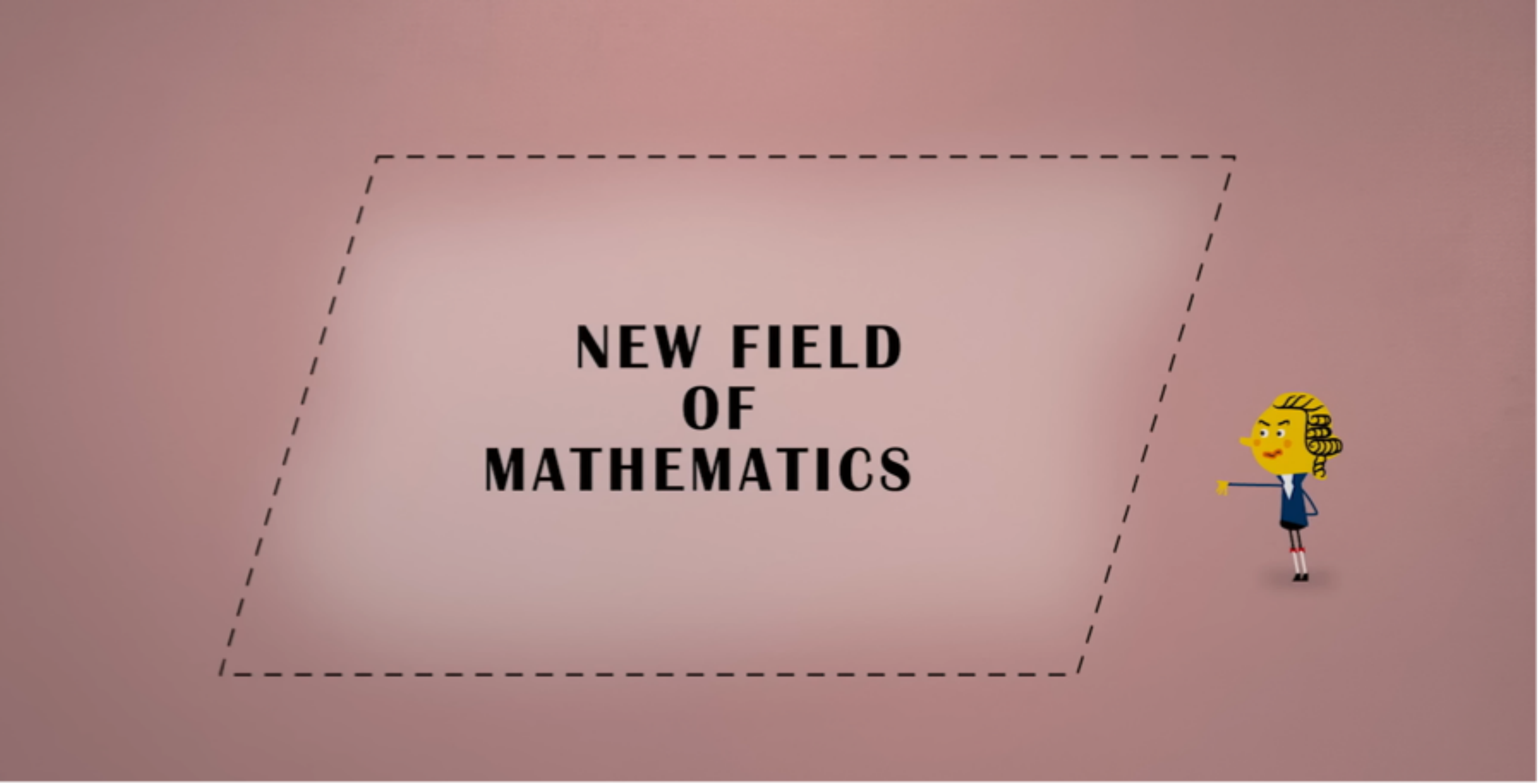




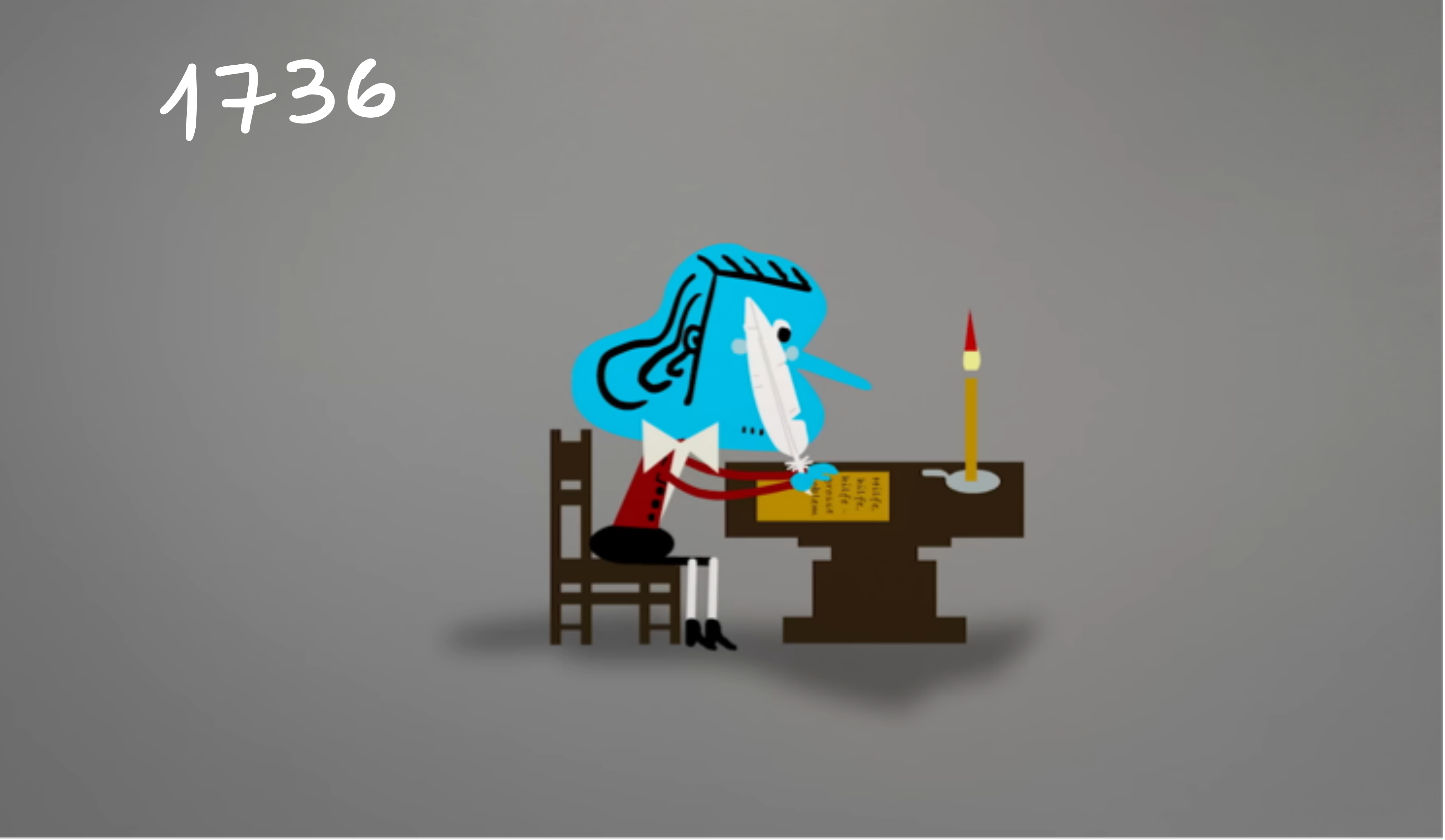
BUT ATTEMPTING TO EXPLAIN WHY...



LED A FAMOUS MATHEMATICIAN...



To invent a new field of mathematics.

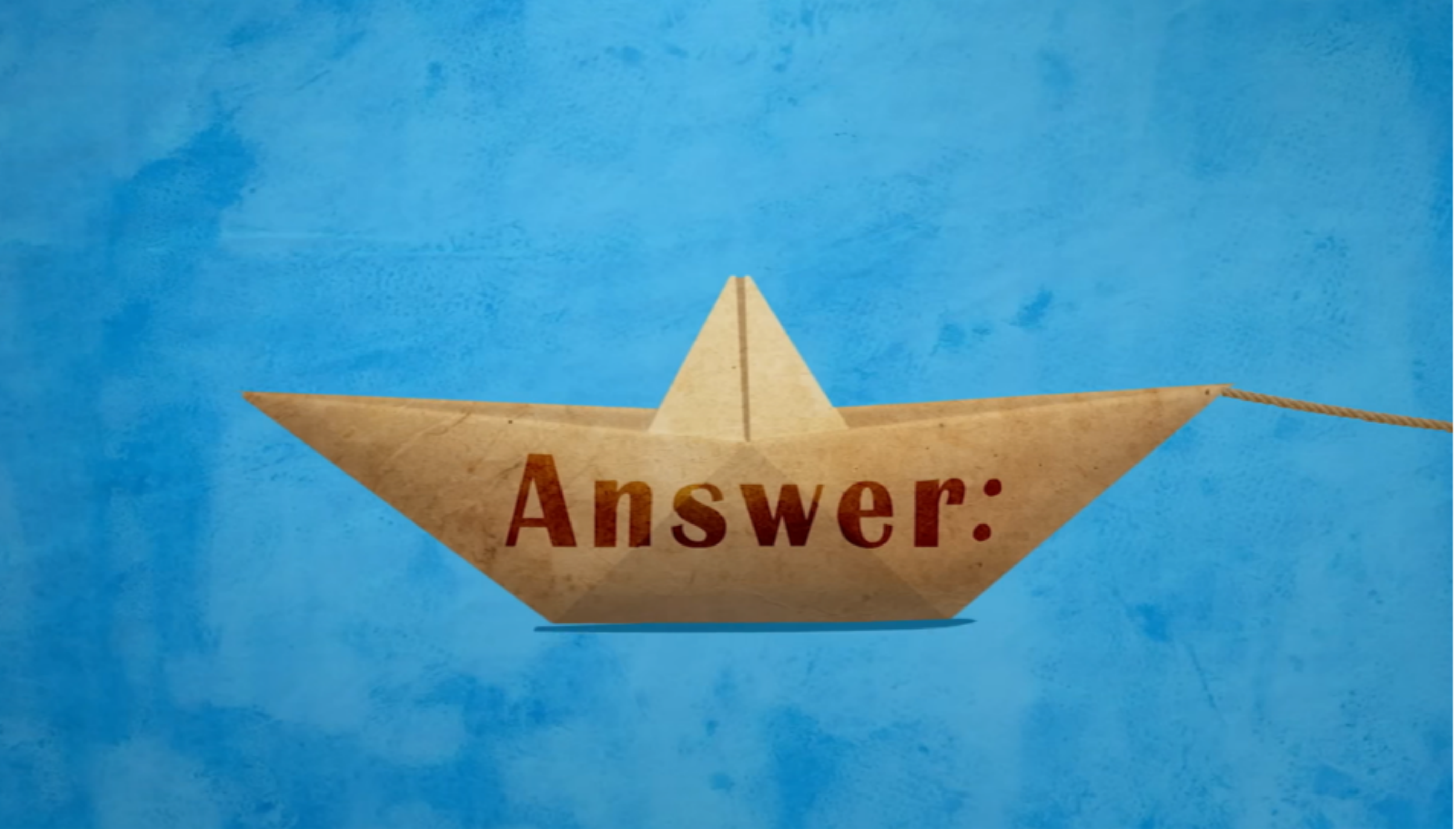


CARL WROTE TO EULER FOR HELP WITH THE PROBLEM.

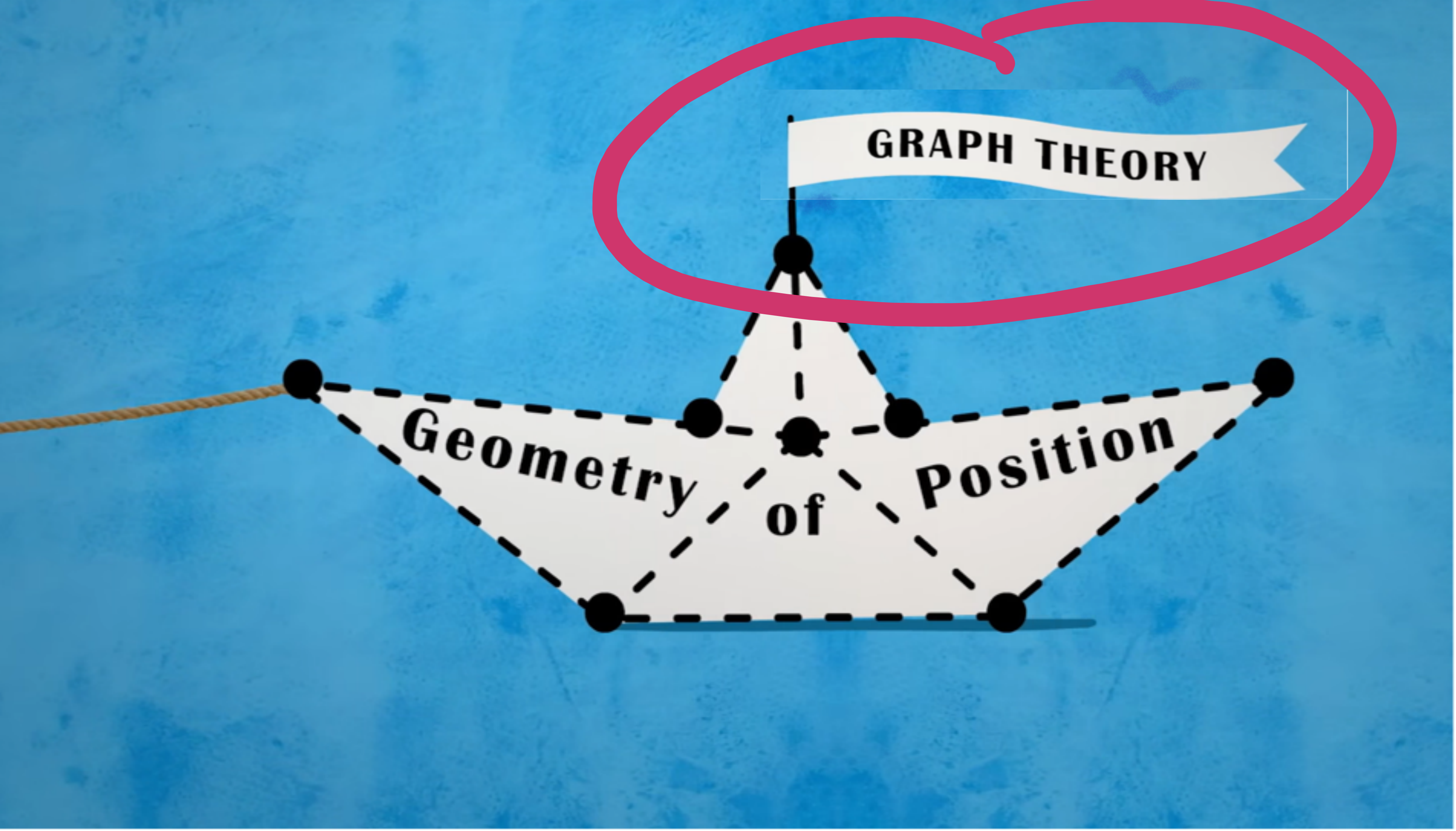
EULER FIRST DISMISSED THE QUESTION AS HAVING NOTHING TO DO WITH MATH.

EULER KEPT THINKING AND THINKING ...

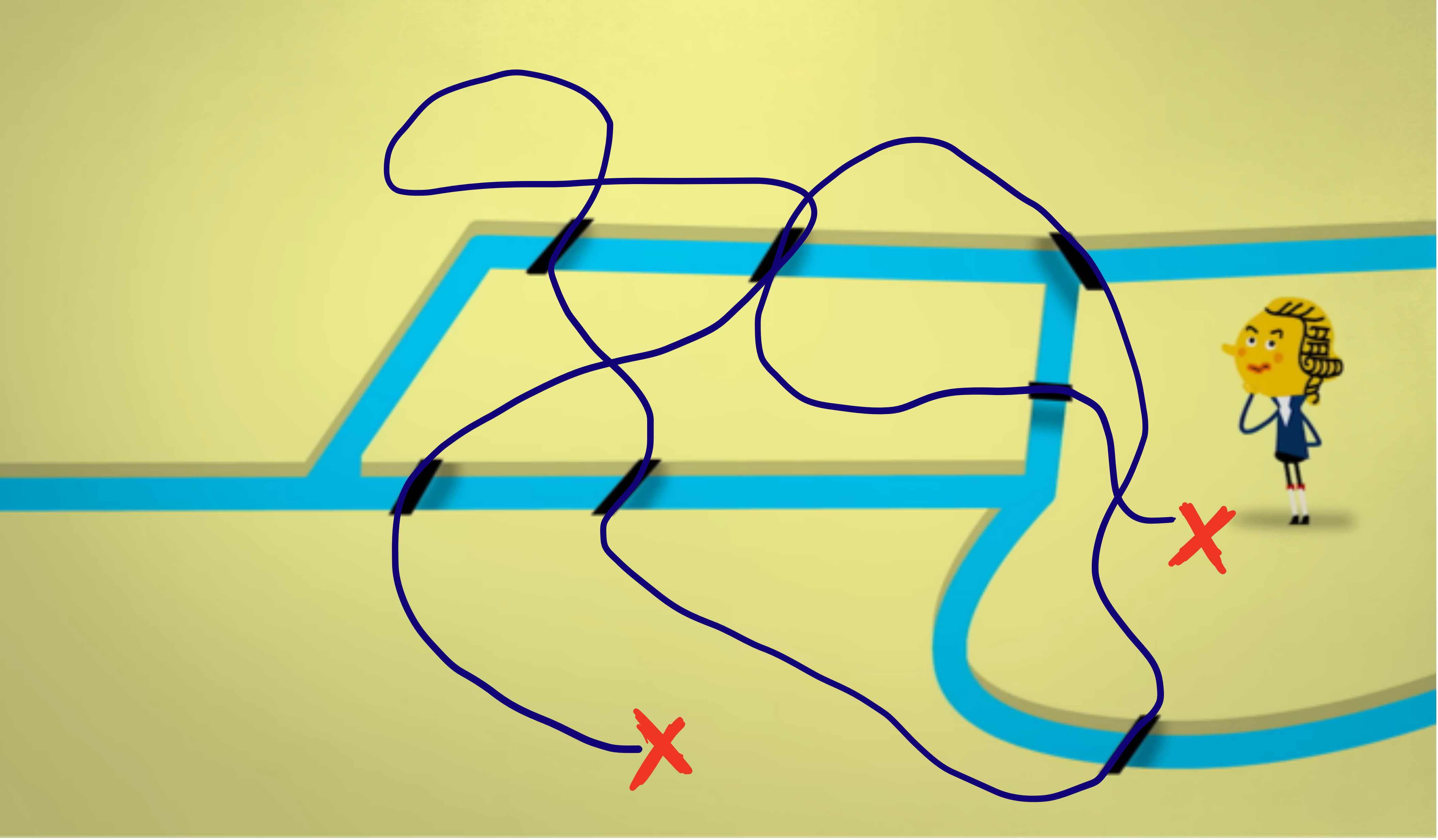
AND IT SEEMED LIKE THERE MIGHT BE SOMETHING THERE ...



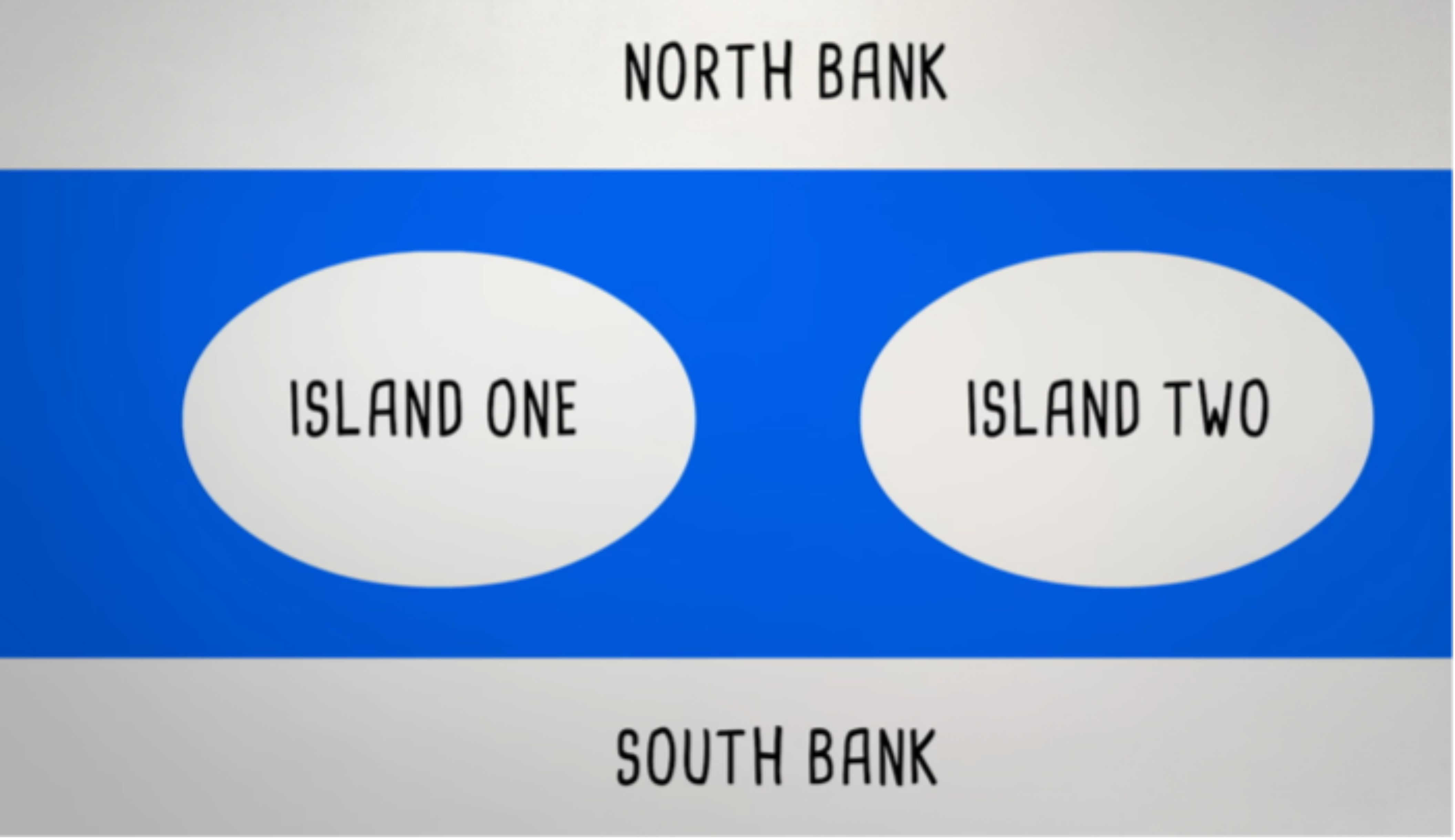
THE ANSWER HE CAME UP WITH HAD TO DO WITH ...



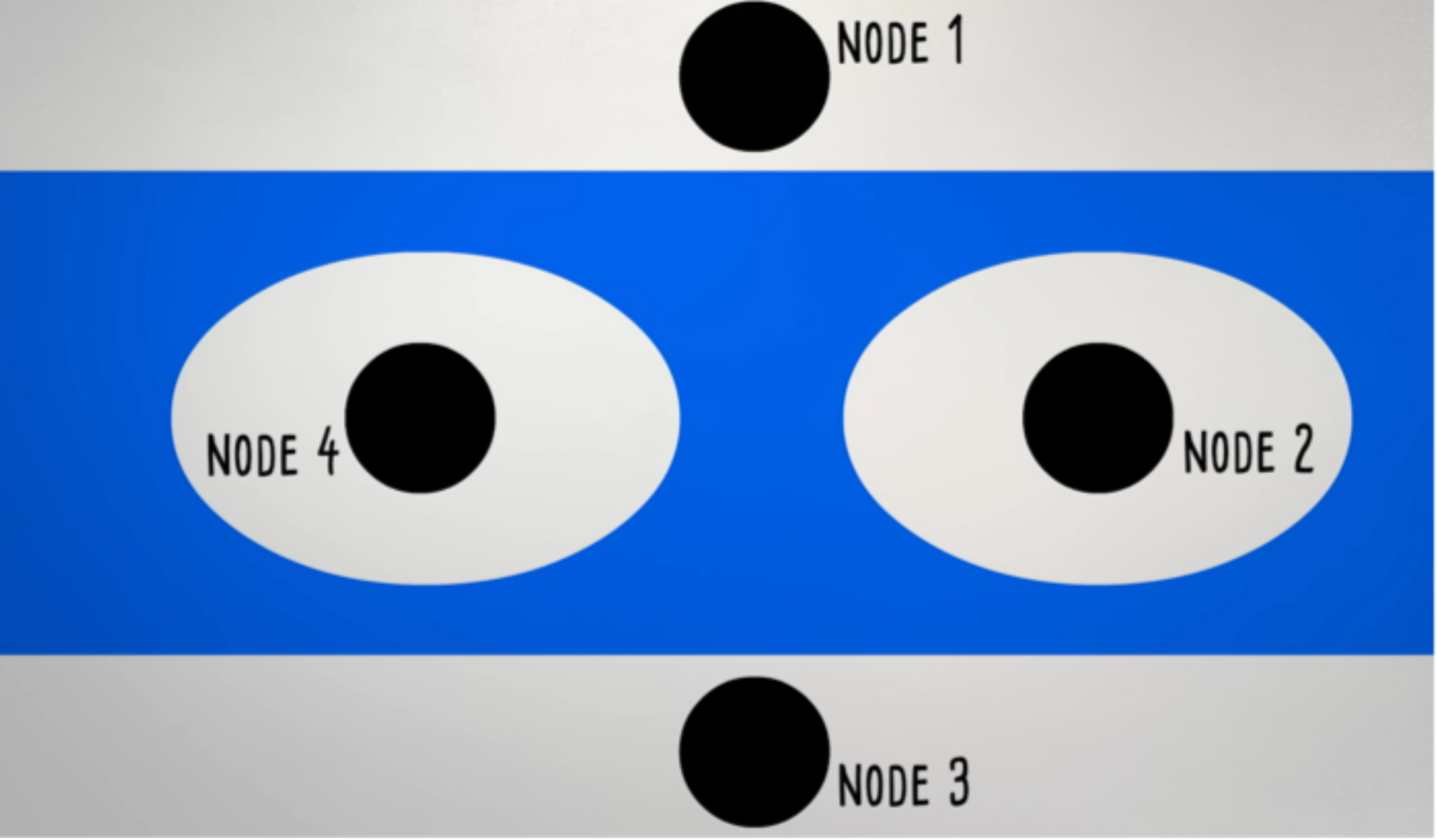
A TYPE OF GEOMETRY ...



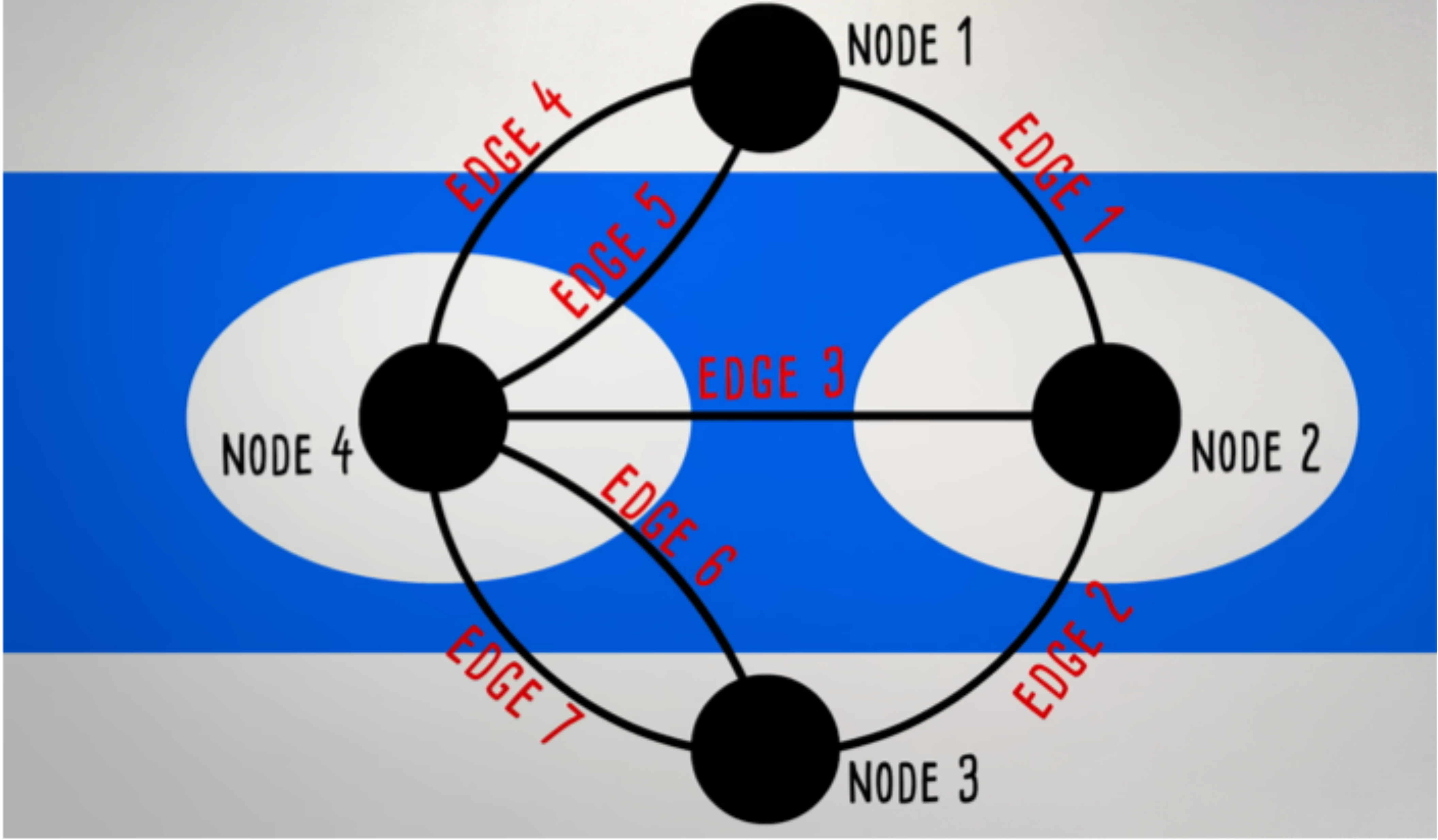
EULER FIGURED OUT THAT IT DOES NOT MATTER WHICH WAY YOU GO WHEN YOU ENTER AND LEAVE AN ISLAND OR RIVERBANK...



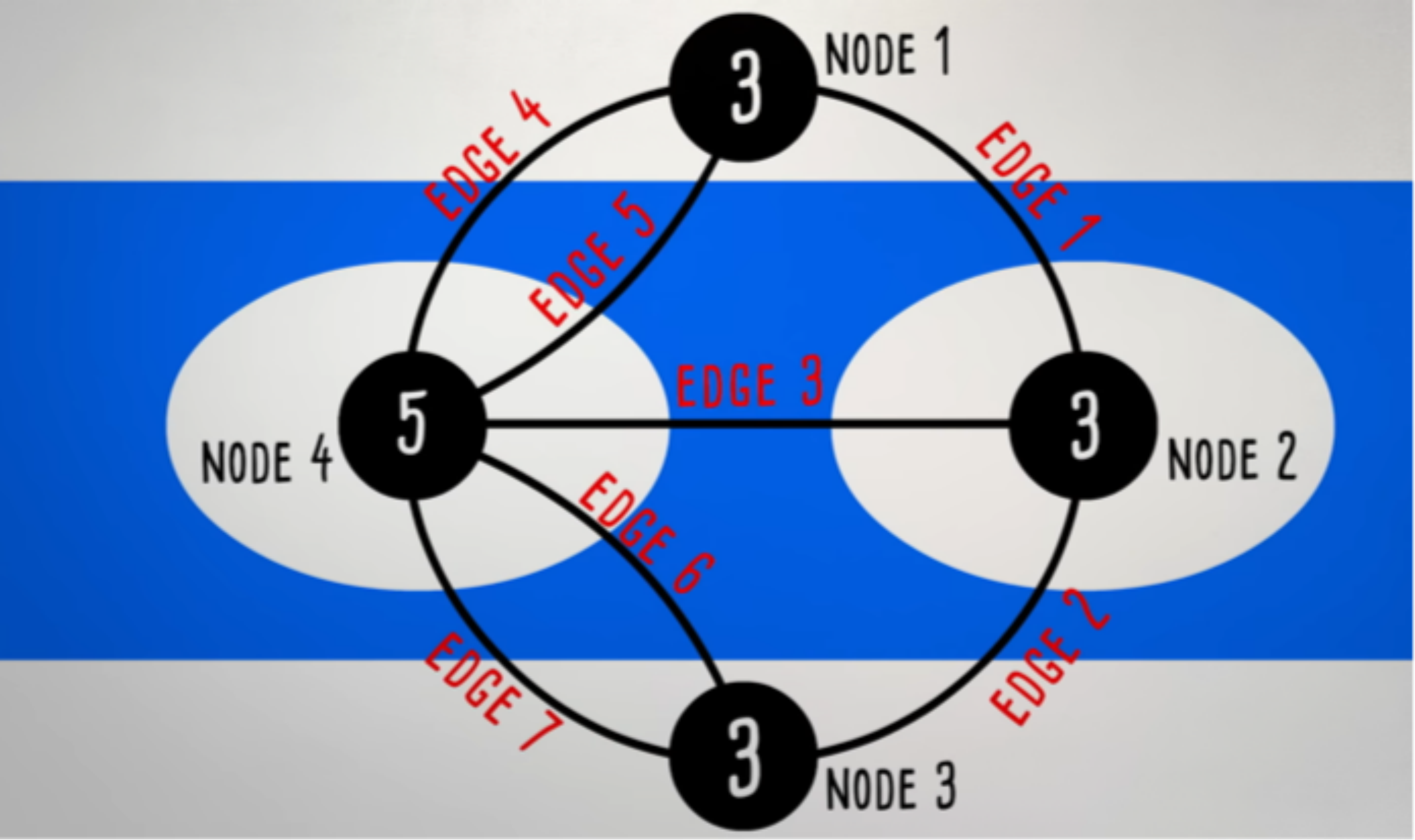
THE MAP COULD BE SIMPLIFIED...



... WITH EACH OF THE FOUR LANDMASSES REPRESENTED AS A SINGLE POINT ...



... WITH LINES BETWEEN THEM TO REPRESENT THE BRIDGES.



THIS SIMPLIFIED GRAPH ALLOWS US TO EASILY COUNT THE DEGREE OF EACH NODE

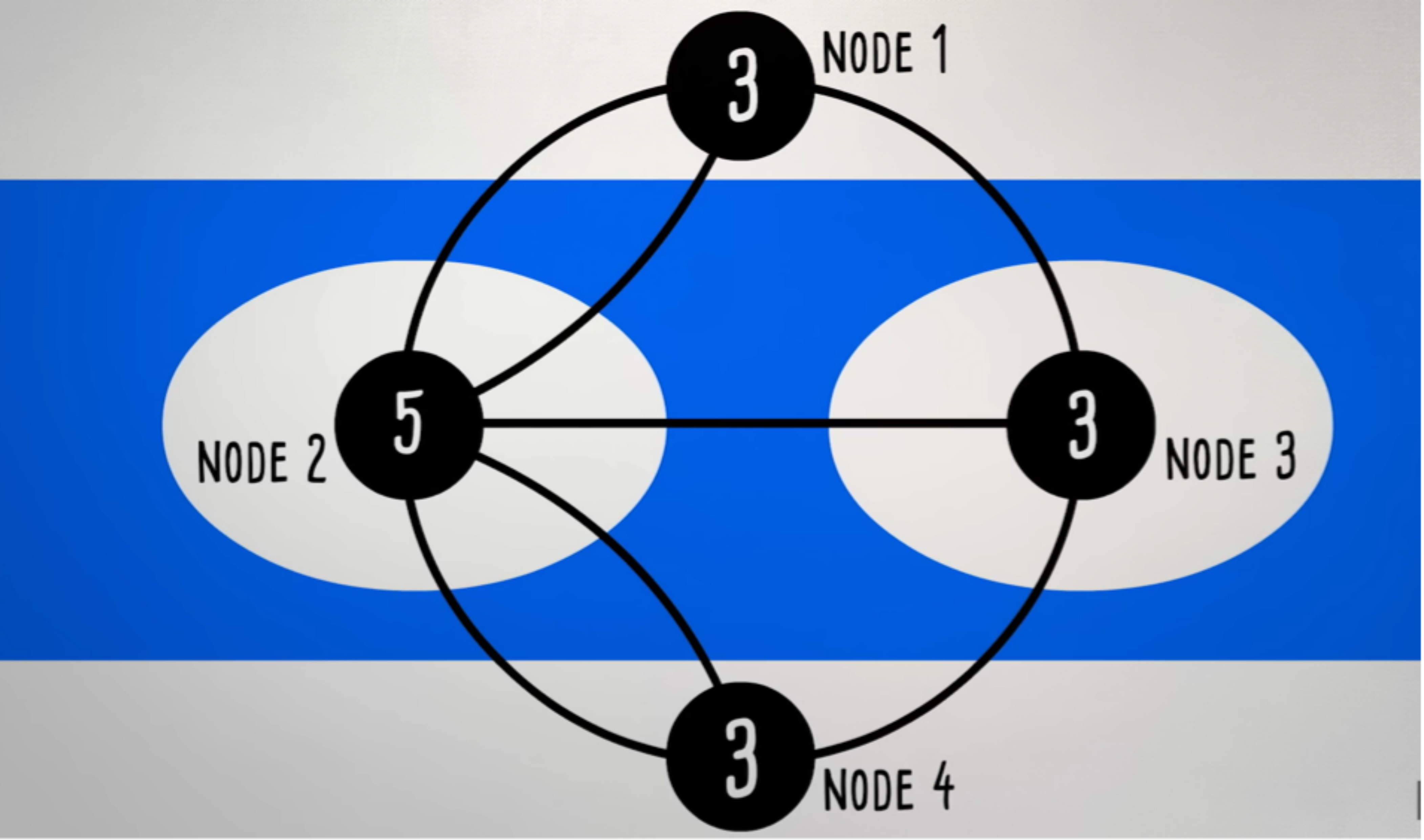
WHY DO THE DEGREES MATTER?

WHEN TRAVELLERS GET TO A LANDMASS USING JUST ONE BRIDGE

THEY WOULD HAVE TO LEAVE IT
VIA A DIFFERENT BRIDGE.

THE NUMBER OF BRIDGES TOUCHING EACH LANDMASS VISITED MUST BE ...

THE ONLY POSSIBLE EXCEPTIONS WOULD BE THE LOCATION OF THE BEGINNING AND END OF THE WALK

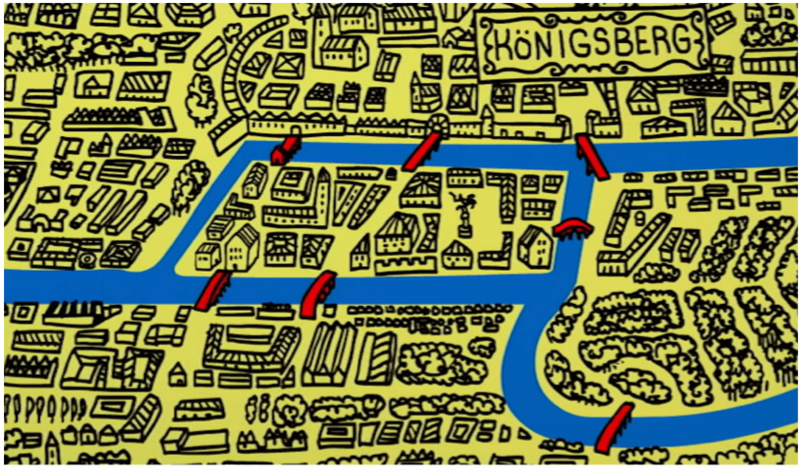


LOOKING AT THE GRAPH, WE OBSERVE THAT ALL FOUR NODES HAVE AN ODD DEGREE.

NO MATTER WHICH PATH IS CHOSEN,
AT SOME POINT, A BRIDGE WILL HAVE TO BE CROSSED TWICE

EULER USED THIS PROOF TO FORMULATE A GENERAL THEORY ...

THE KÖNIGSBERG BRIDGE PROBLEM



BECOMES MORE INTERSTING LATER...

MEANWHILE, IT SUGGESTS OUR BASIC DEFINITION OF A GRAPH

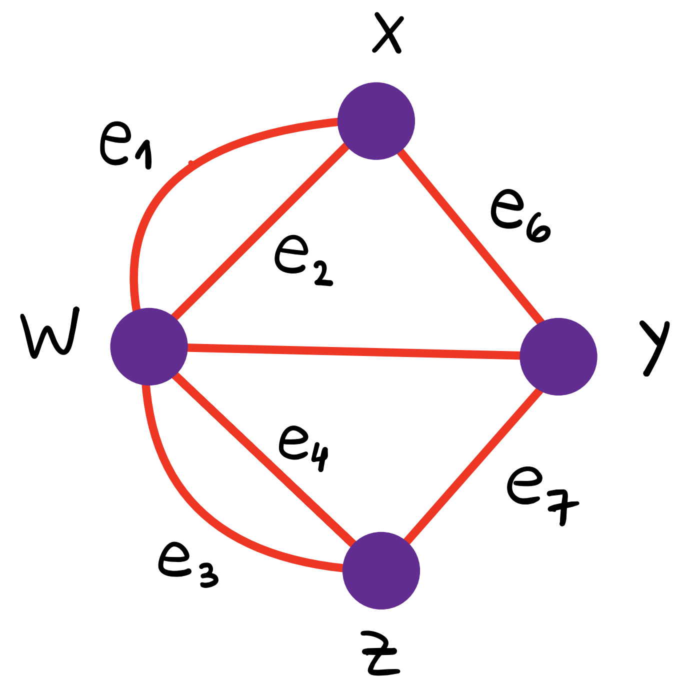
DEFINITION

A GRAPH G IS A TRIPLE CONSISTING OF A VERTEX SET $V(G)$, AN EDGE SET $E(G)$, AND A RELATION THAT ASSOCIATES WITH EACH EDGE TWO VERTICES (NOT NECESSARILY DISTINCT) CALLED ITS ENDPOINTS.

REMARK

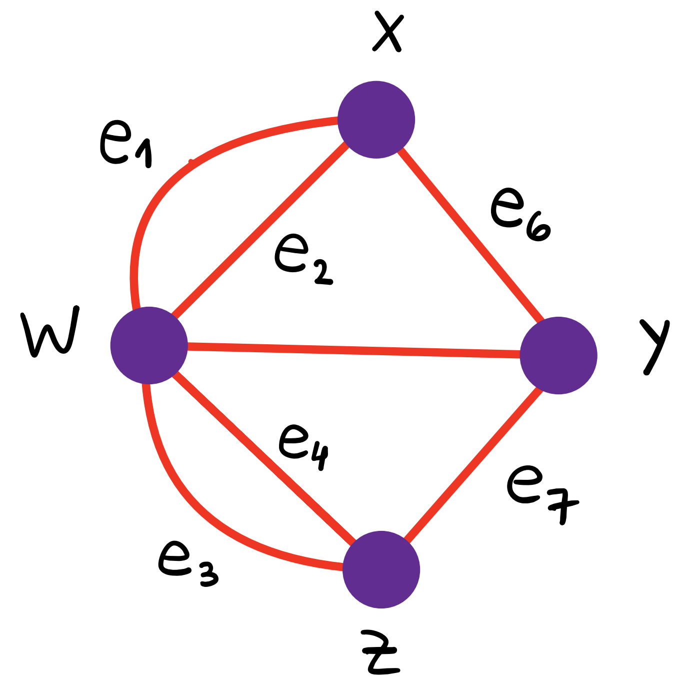
WE DRAW A GRAPH ON A PAPER BY PLACING EACH VERTEX AT A POINT AND REPRESENTING EACH EDGE BY A CURVE JOINING THE LOCATIONS OF ITS ENDPOINTS.

THE KÖNIGSBERG BRIDGE PROBLEM



REMARK

THIS GRAPH G HAS VERTEX SET $V(G) = \{x, y, z, w\}$, THE EDGE SET IS $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, AND THE ASSIGNMENT OF ENDPOINTS TO EDGES CAN BE READ FROM THE PICTURE.



Remark

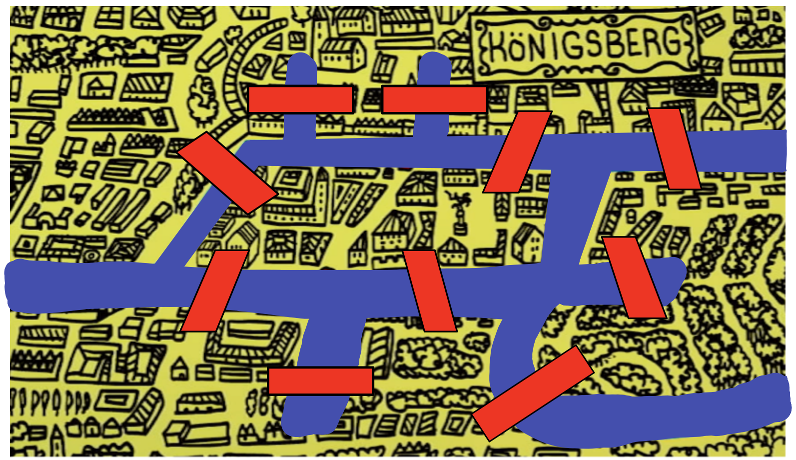
Note that edges e_1 and e_2 have the same endpoints, as do e_3 and e_4. Also if we had a bridge over an inlet, then its ends would be in the same land mass.

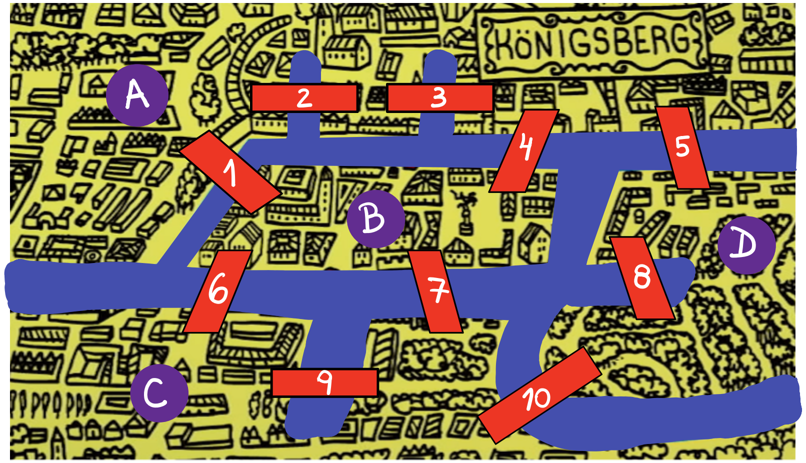
We have appropriate terms for these types of edges in graphs ...

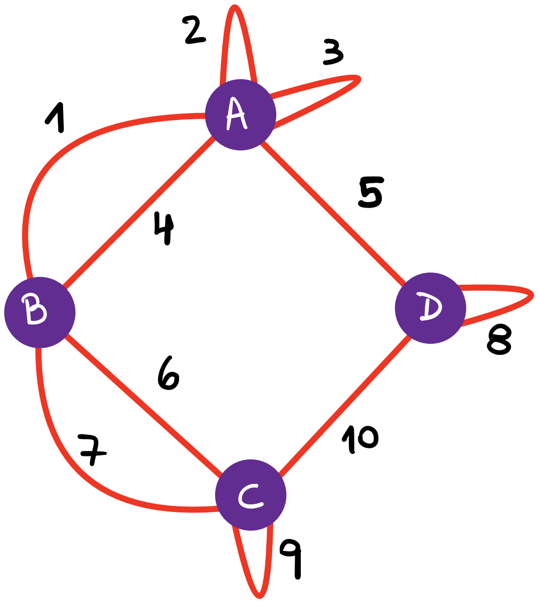
DEFINITION

A **LOOP** IS AN EDGE WHOSE ENDPOINTS ARE EQUAL.

MULTIPLE EDGES ARE EDGES HAVING THE SAME PAIR OF ENDPOINTS.







Definition

A **simple graph** is a graph having no loops or multiple edges.

Remark

We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with endpoints u and v.

Definition

Two vertices are **adjacent** or **neighbours** when they are the endpoints of an edge.

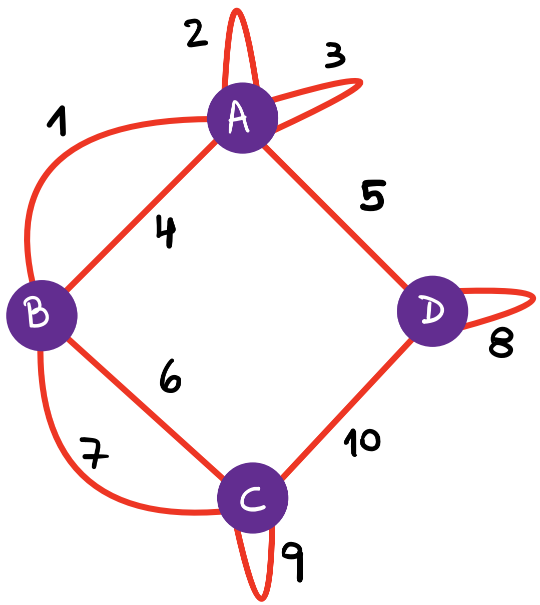
Definition

A graph is **finite** if its vertex set and edge set are finite.

Remark

Every graph mentioned in this lecture is finite and simple.

GRAPH G



$V(G) = \{A, B, C, D\}$

$E(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

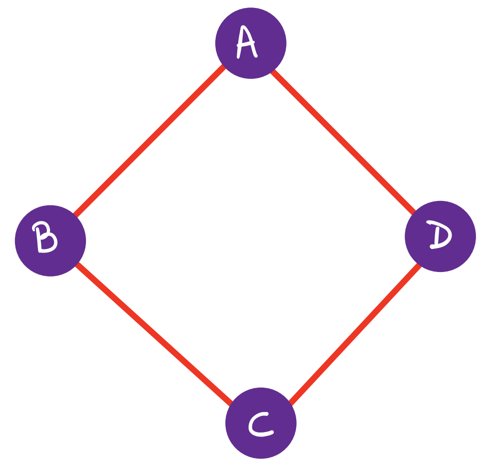
VERTICES B AND C ARE ADJACENT

6 AND 7 ARE MULTIPLE EDGES

2 AND 3 ARE LOOPS.

G IS FINITE BUT NOT SIMPLE.

GRAPH H



$V(H) = \{A, B, C, D\}$

$E(H) = \{\{A,B\}, \{B,C\}, \{C,D\}, \{A,D\}\}$

B AND D ARE NEIGHBOURS OF A.

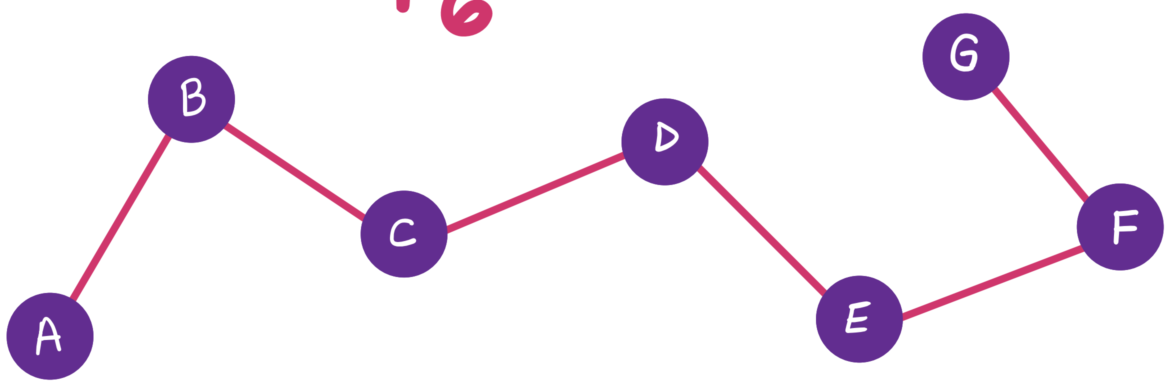
G IS FINITE AND SIMPLE.

SOME SPECIAL SIMPLE GRAPHS

DEFINITION

A PATH is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

P_6



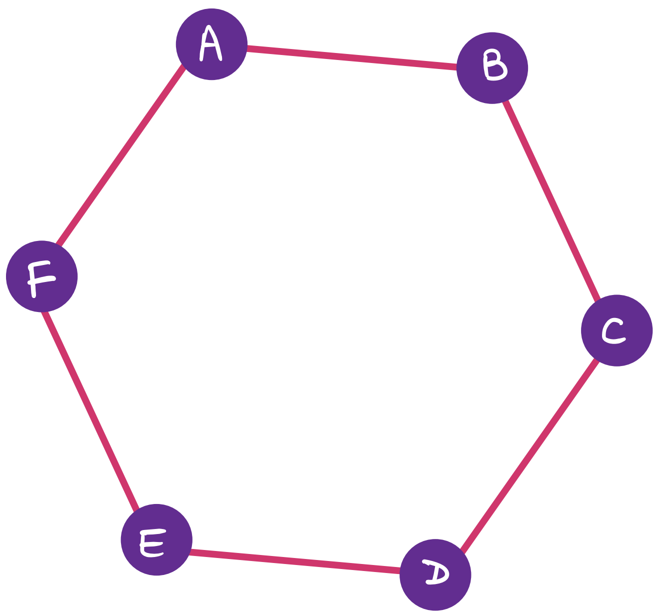
A PATH OF LENGTH 6

SOME SPECIAL SIMPLE GRAPHS

DEFINITION

A CYCLE IS A GRAPH WITH AN EQUAL NUMBER OF VERTICES AND EDGES WHOSE VERTICES CAN BE PLACE AROUND A CIRCLE SO THAT TWO VERTICES ARE ADJACENT IF AND ONLY IF THEY APPEAR CONSECUTIVELY ALONG THE CYCLE.

C_6

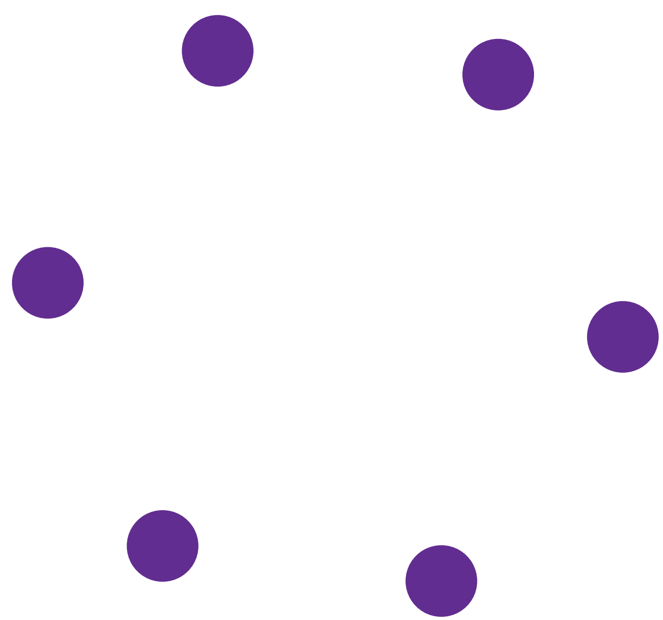


A CYCLE OF LENGTH 6

SOME SPECIAL SIMPLE GRAPHS

DEFINITION

FOR EACH NATURAL NUMBER m, THE EDGELESS GRAPH S_m IS THE GRAPH WITH m VERTICES AND ZERO EDGES.

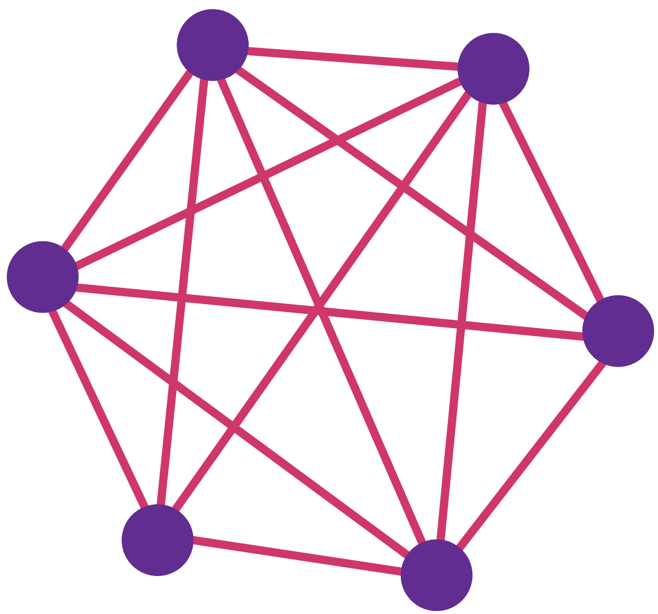


S_6

Some Special Simple Graphs

Definition

For each natural number m, the complete graph K_m is the graph with m vertices in which every pair of distinct vertices is connected by a unique edge.

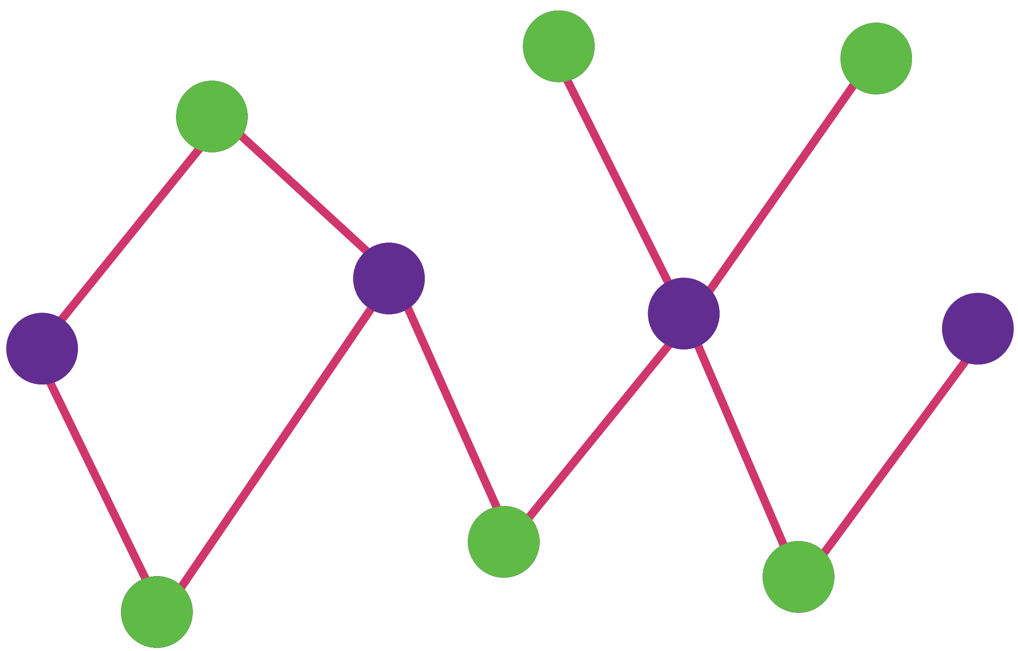


K_6

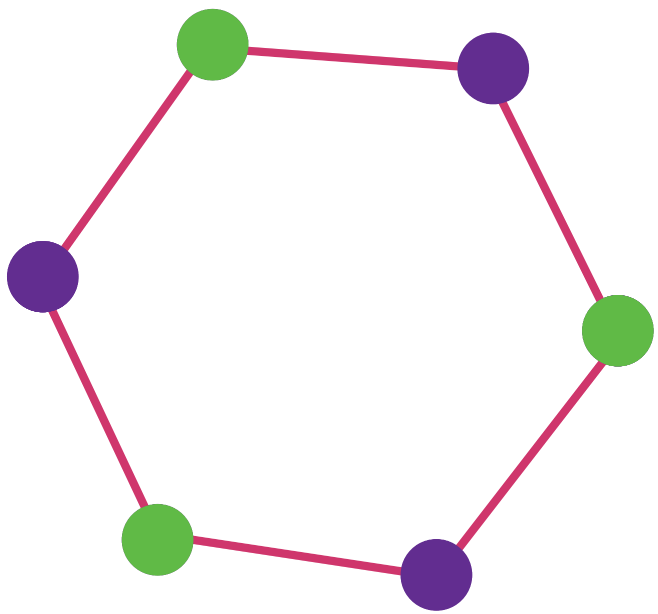
SOME SPECIAL SIMPLE GRAPHS

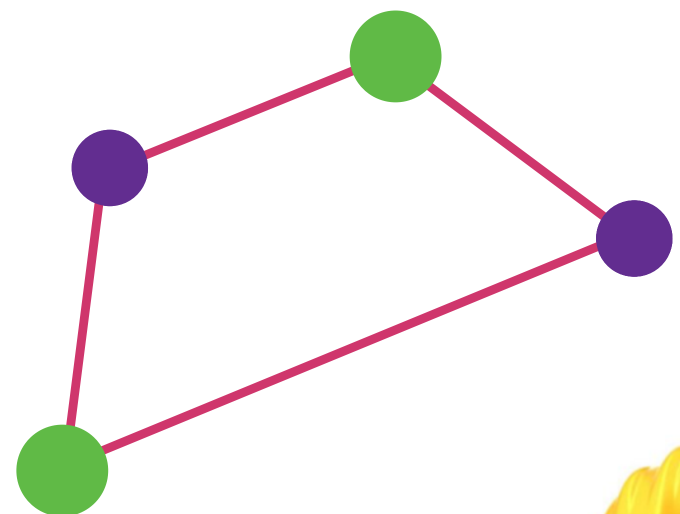
DEFINITION

A GRAPH IS BIPARTITE IF WE ONLY REQUIRE TWO COLORS FOR COLORING ITS VERTICES IN SUCH A WAY THAT ADJACENT VERTICES RECEIVED DIFFERENT COLORS.



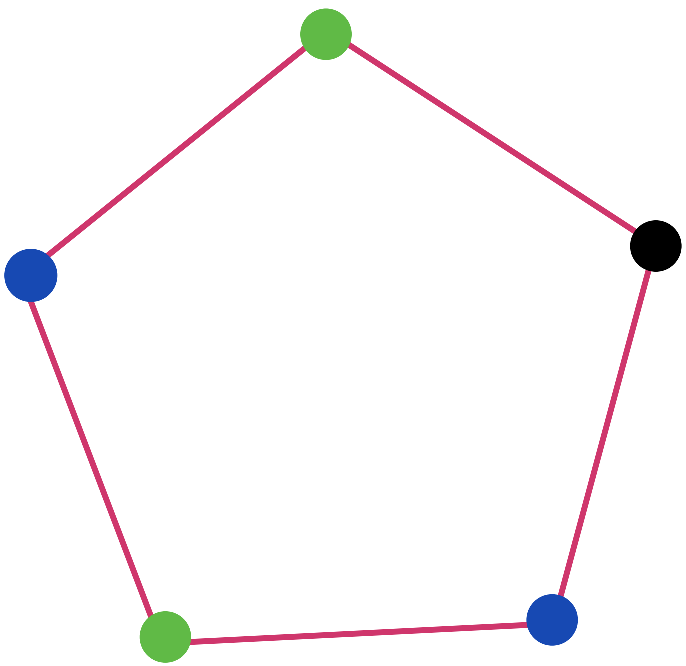
Cycles of Even Length

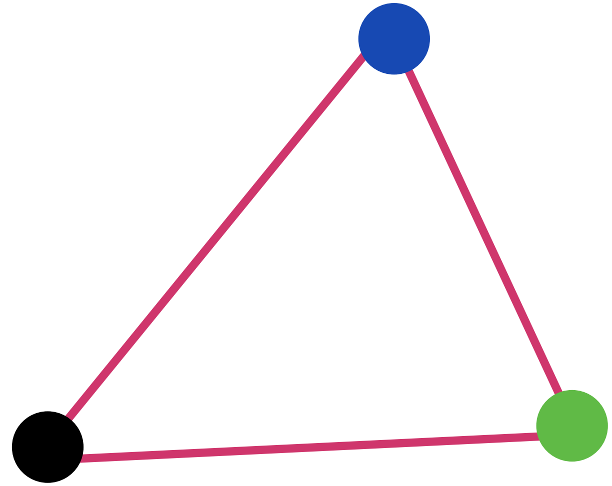




Bipartite

CYCLES OF ODD LENGTH

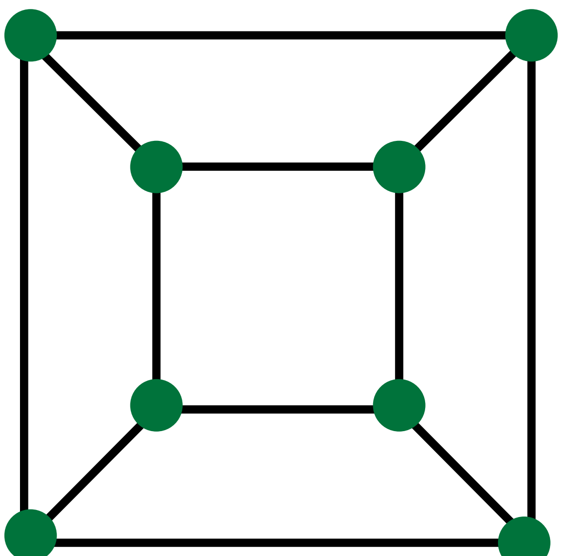


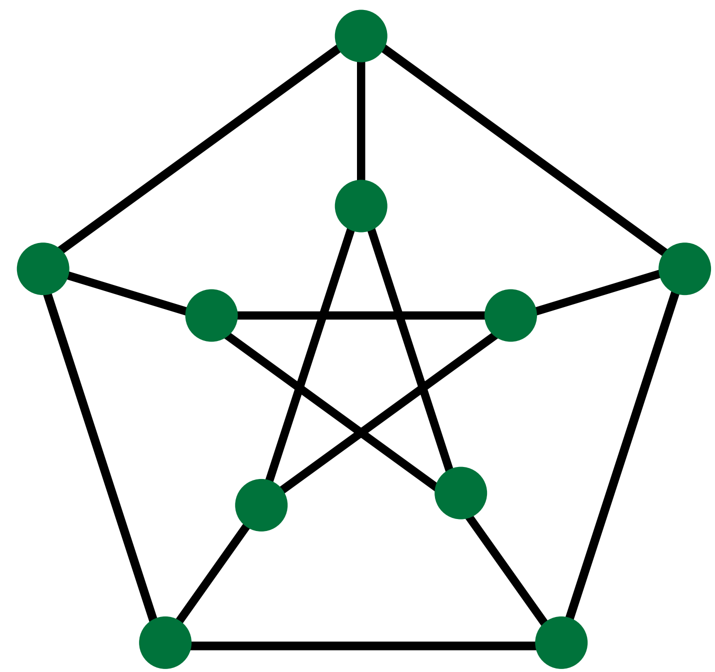


NON - BIPARTITE

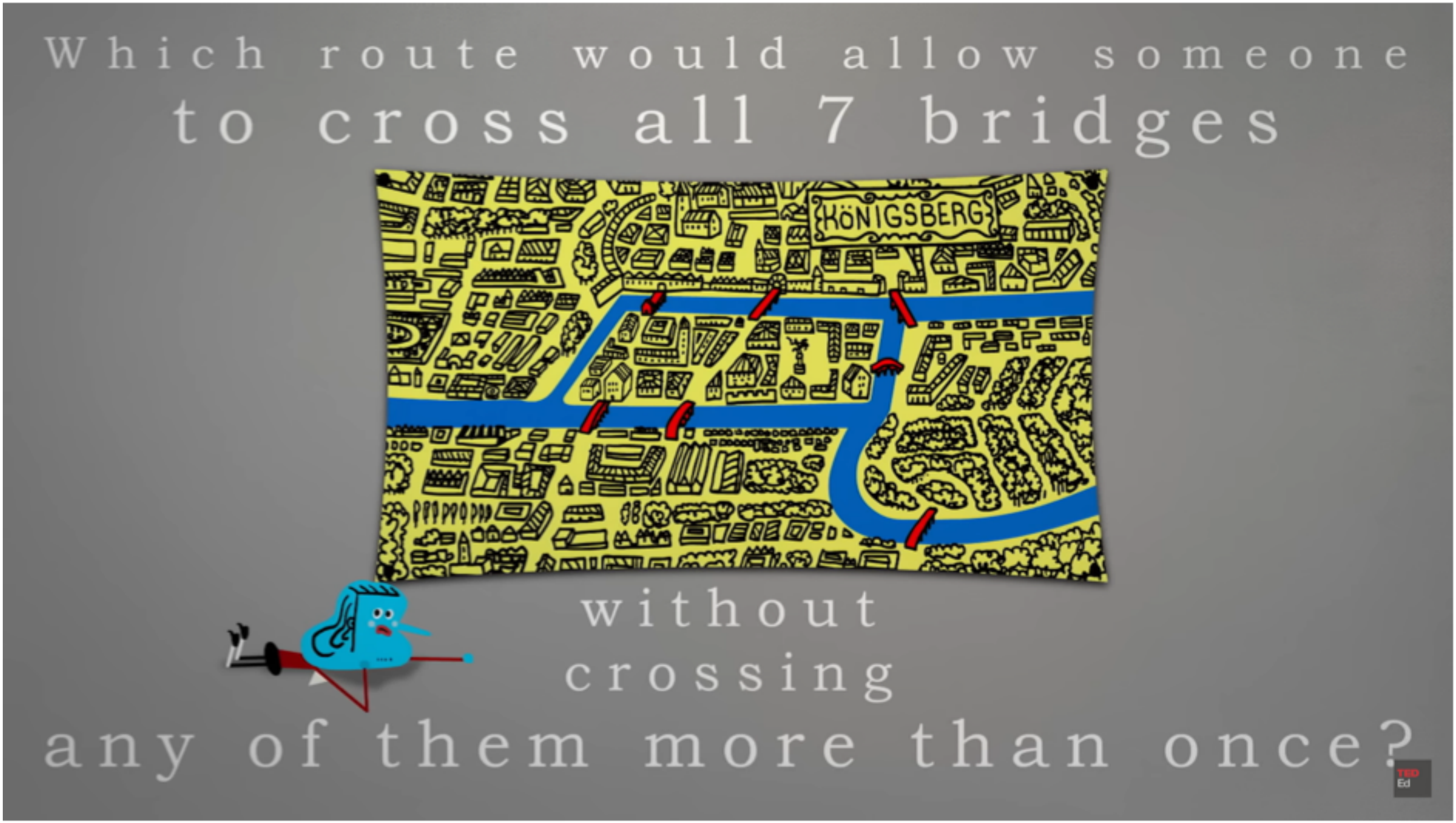
THEOREM (KÖNIG, 1936)

A GRAPH IS BIPARTITE IF AND ONLY IF IT HAS NO CYCLES OF ODD LENGTH.

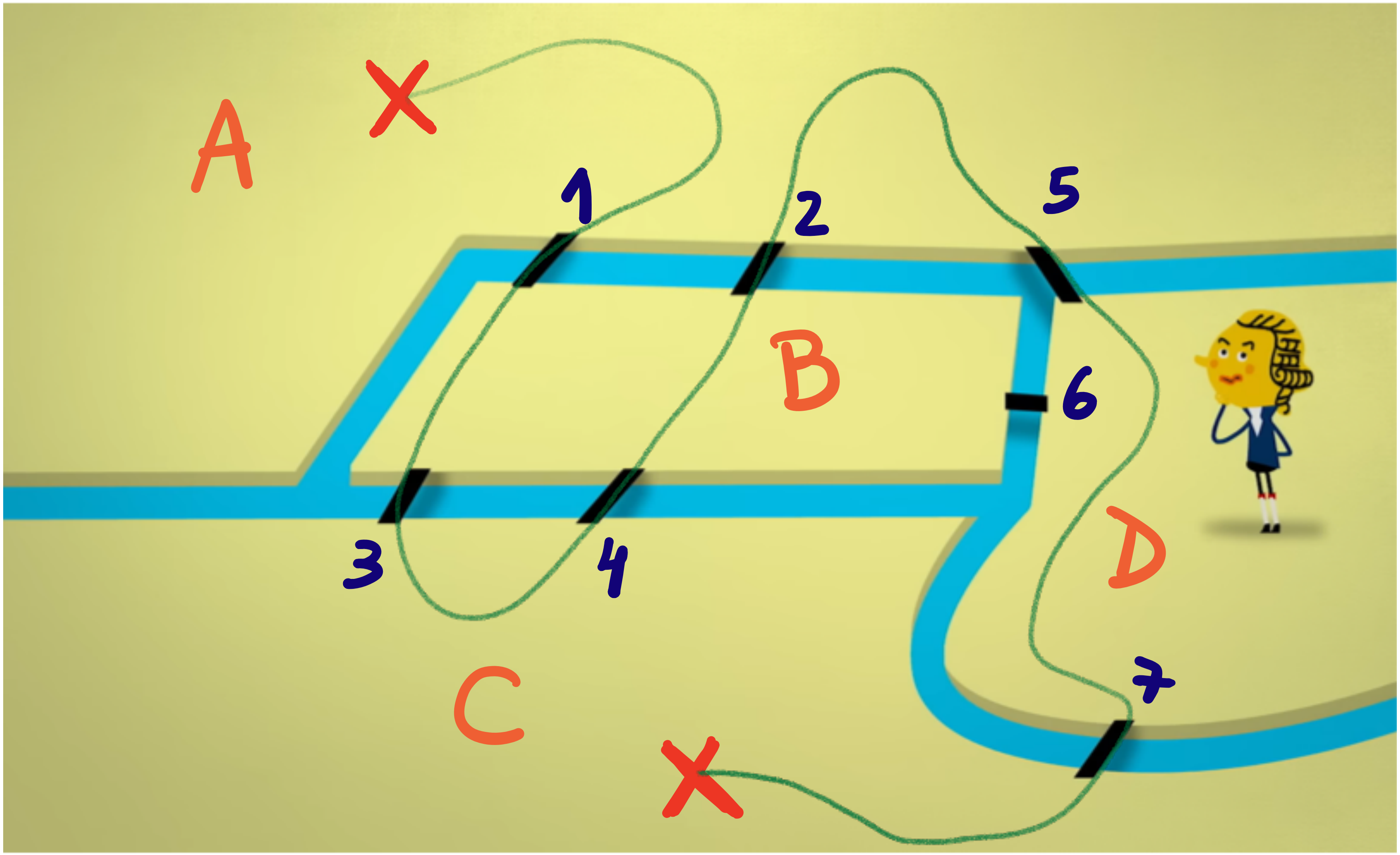




THE KÖNIGSBERG BRIDGE PROBLEM



A WALK THROUGH THE CITY



[A, 1, B, 3, C, 4, B, 2, A, 5, D, 7, C]

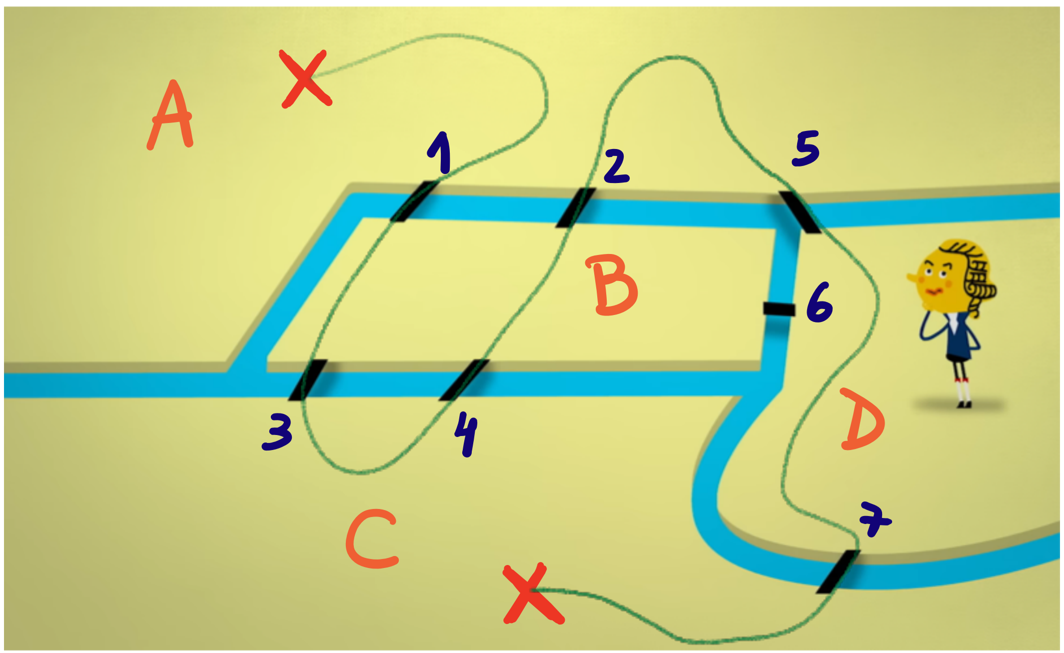
DEFINITION

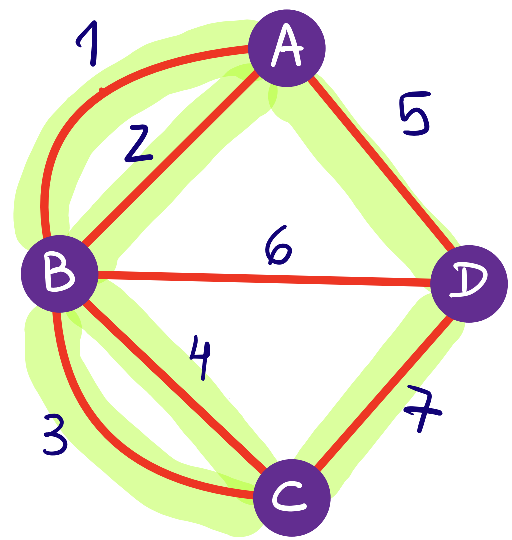
A **WALK** in a graph is a sequence of vertices and edges,

$$[v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_k, e_k, v_{k+1}]$$

such that the endpoints of edge e_i are v_i and v_{i+1} $(1 \leq i \leq k)$.

If $v_1 = v_{k+1}$ the walk is a **closed walk** or a **circuit**.





[A, 1, B, 3, C, 4, B, 2, A, 5, D, 7, C]

DEFINITION

A GRAPH IS CONNECTED WHENEVER THERE EXIST A WALK CONNECTING ANY TWO VERTICES.

DEFINITION

THE DEGREE OF A VERTEX IS THE NUMBER OF TIMES THAT AN EDGE TOUCHES THE VERTEX.

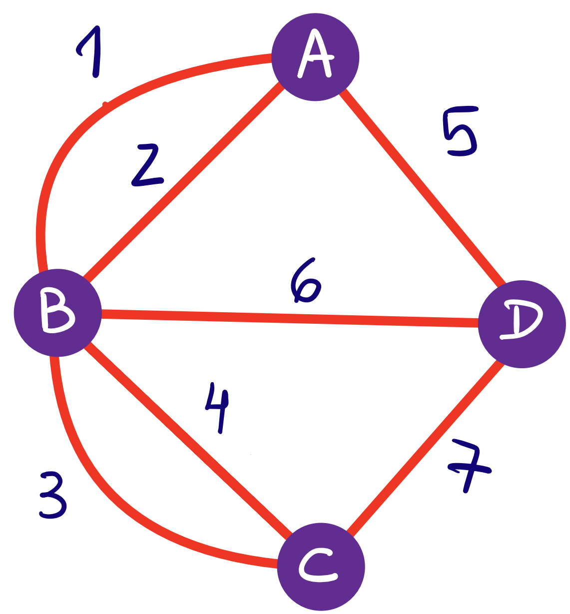
OUR GRAPH IS CONNECTED.

THE DEGREE OF A IS 3.

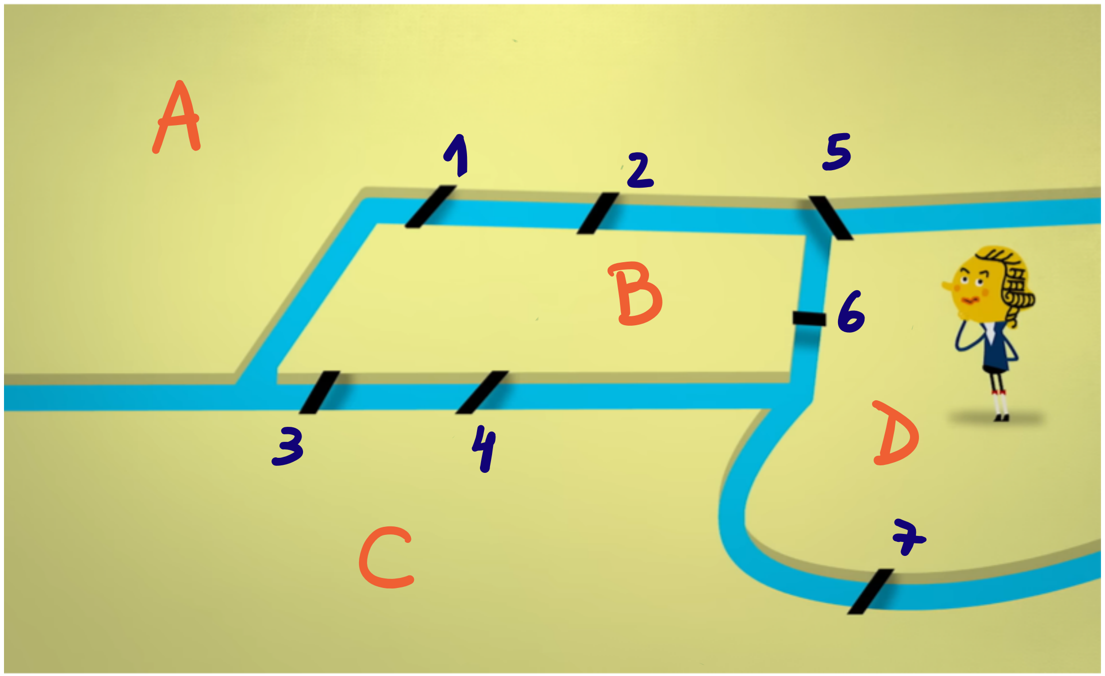
THE DEGREE OF B IS 5.

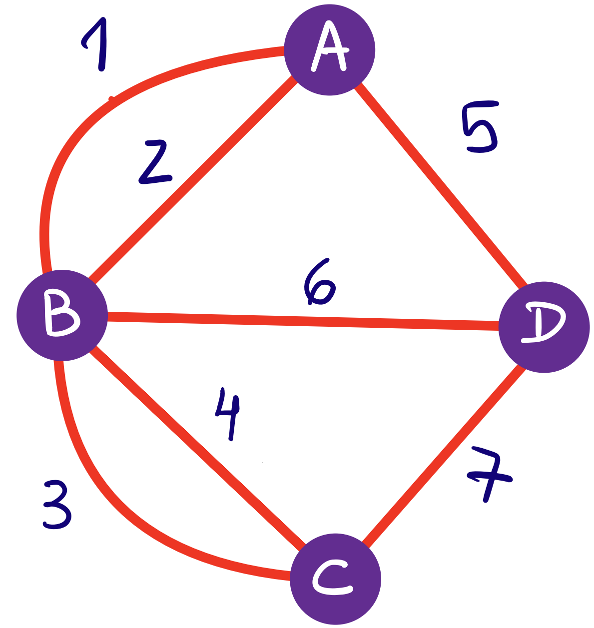
THE DEGREE OF C IS 3.

THE DEGREE OF D IS 3.



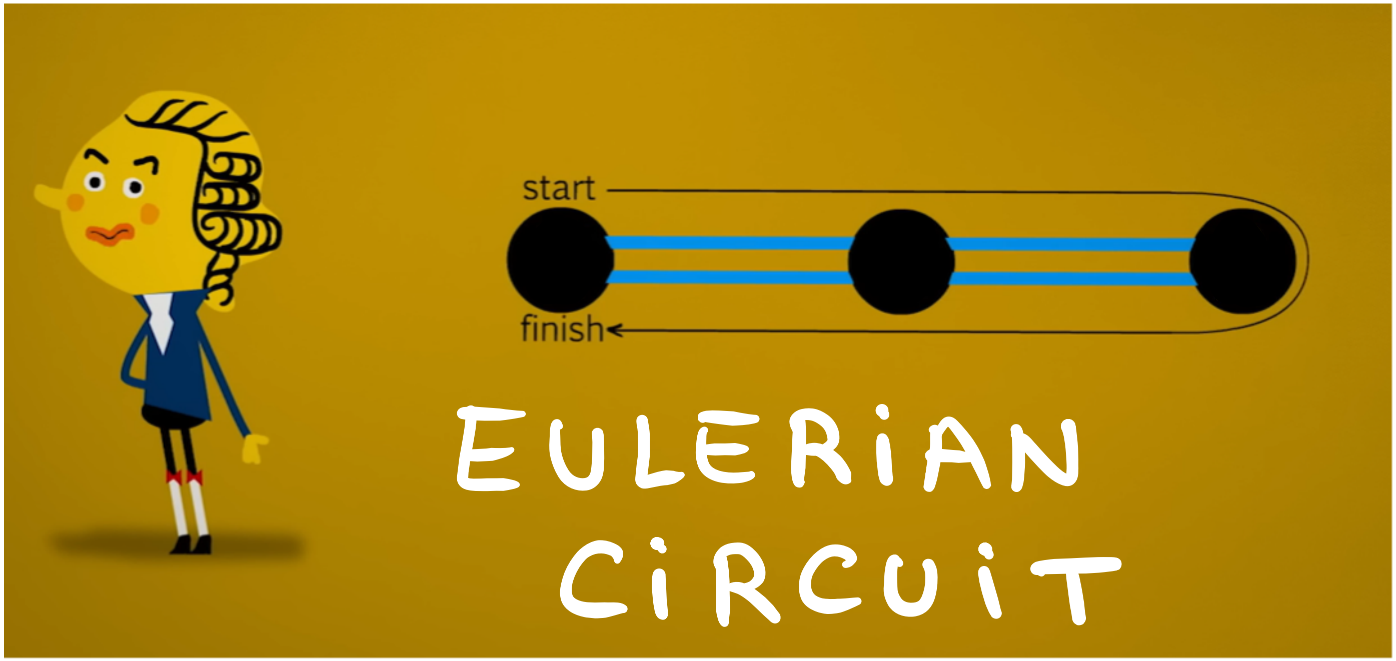
THE KÖNIGSBERG BRIDGE PROBLEM

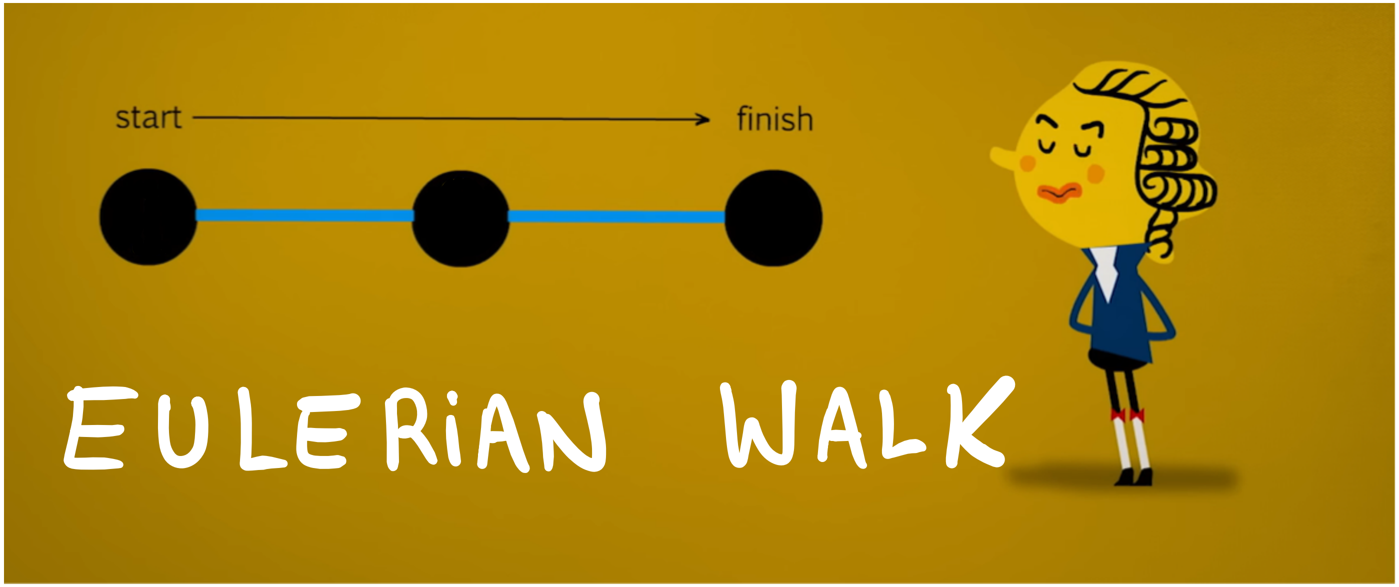




A SUCESSFUL WALK IN KÖNIGSBERG CORRESPOND TO A WALK IN THE GRAPH IN WHICH EVERY EDGE IS USED EXACTLY ONCE.

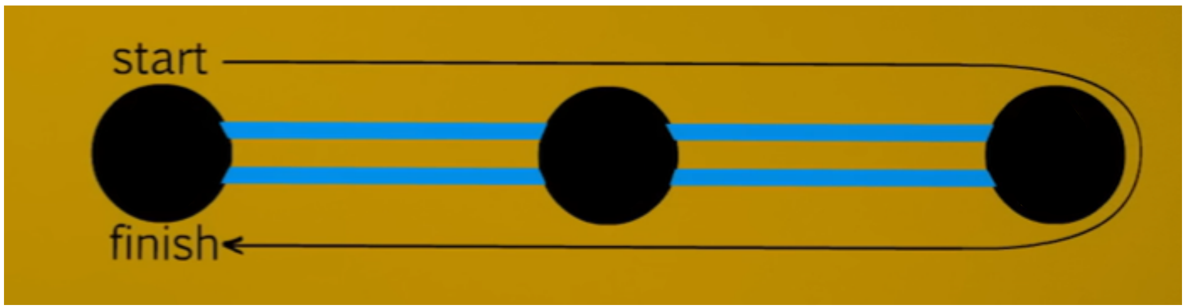
A WALK IN A GRAPH IN WHICH EVERY EDGE IS USED EXACTLY ONCE ...



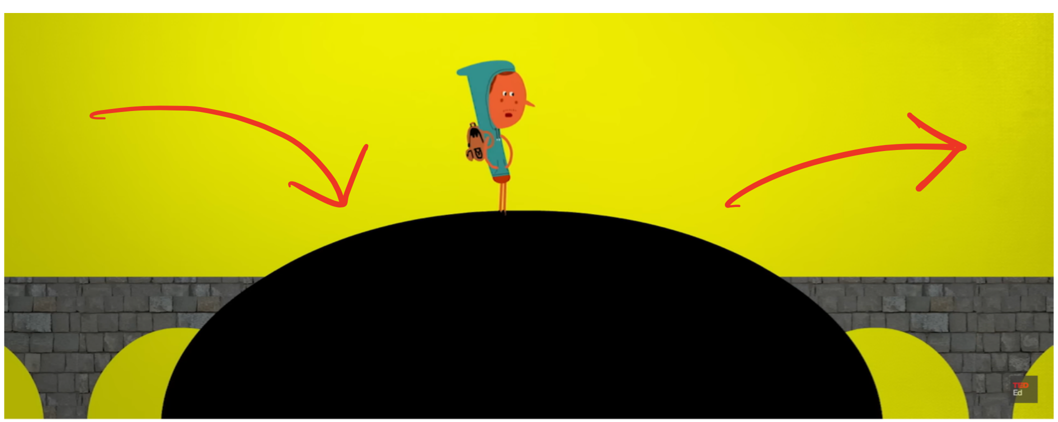


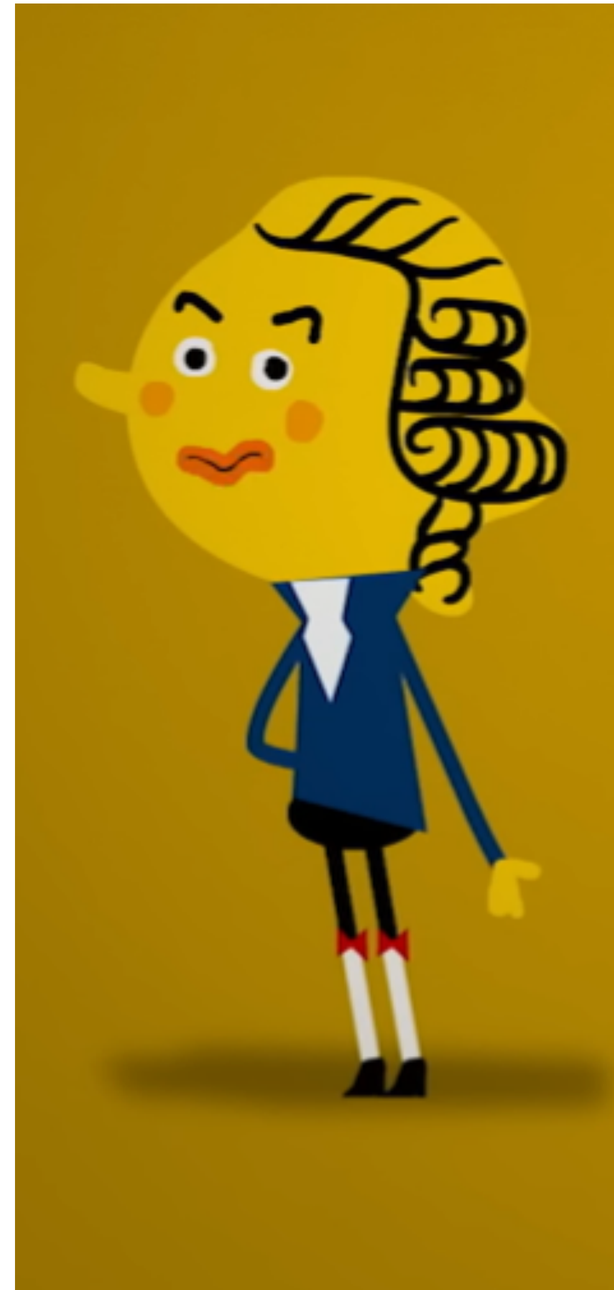
IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

EULERIAN CIRCUIT



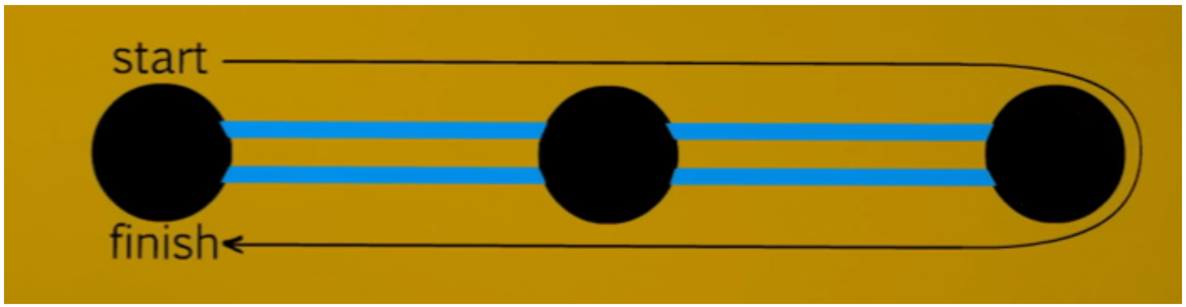
AT EVERY VERTEX OTHER THAN THE COMMON STARTING AND ENDING POINT, WE COME INTO THE VERTEX ALONG ONE EDGE AND GO OUT ALONG ANOTHER (THIS CAN HAPPEN MORE THAN ONCE).



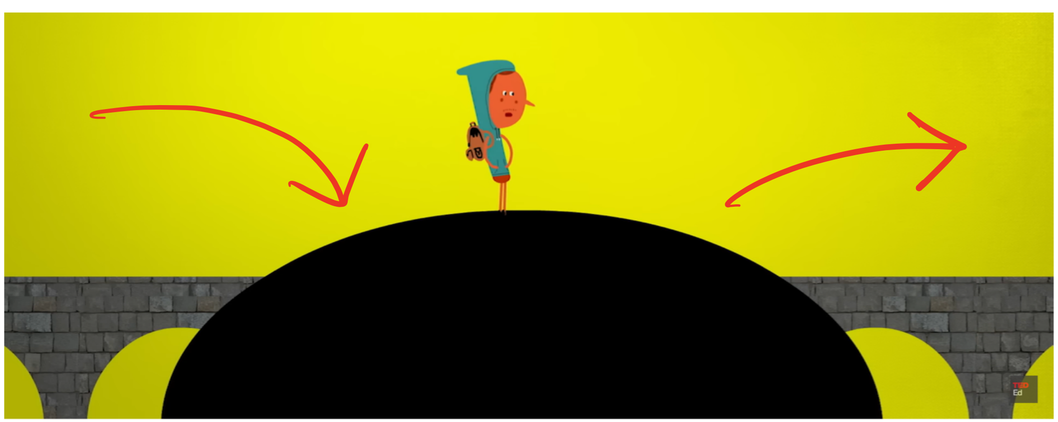


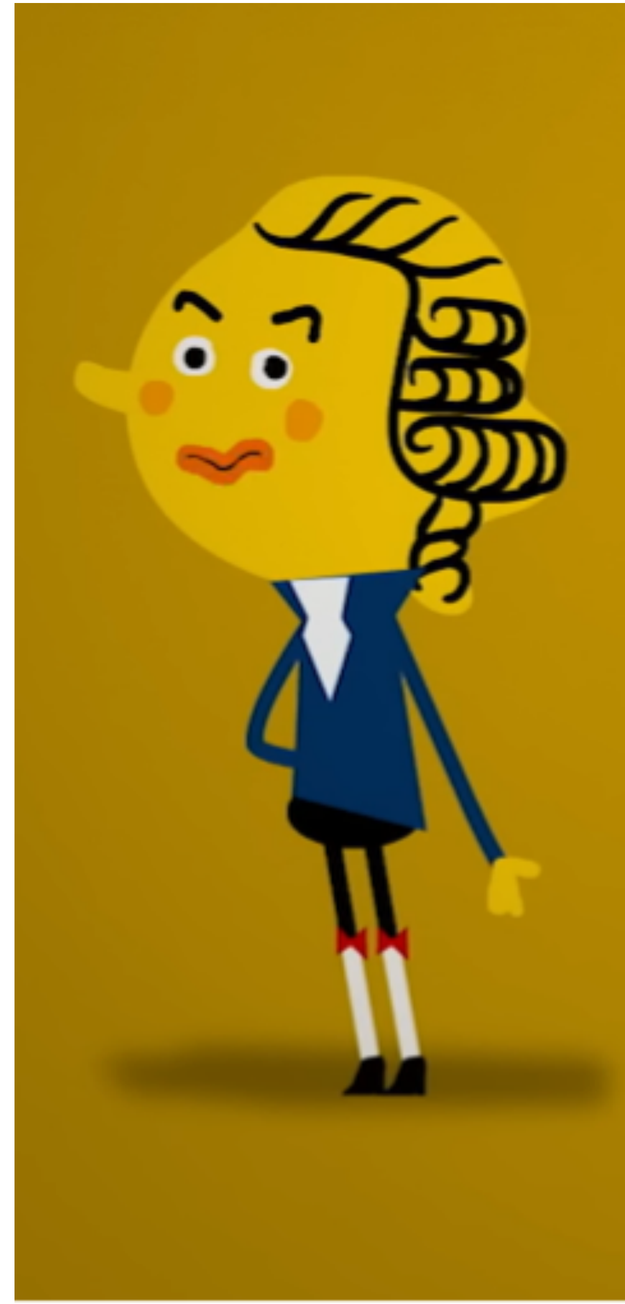
IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

EULERIAN CIRCUIT



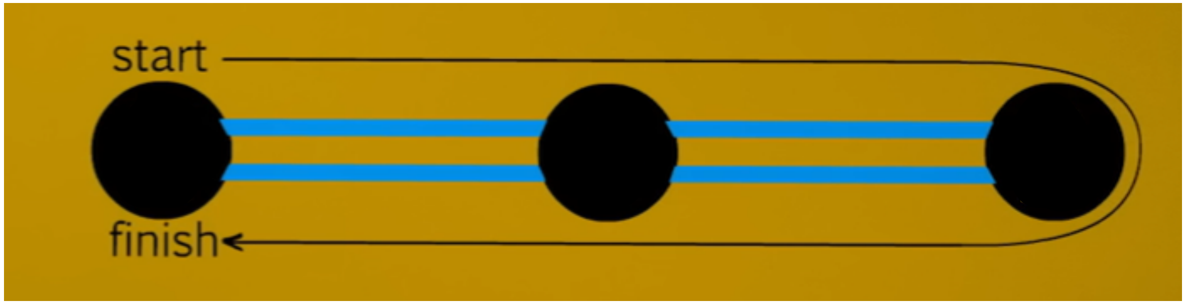
SINCE WE CANNOT USE EDGES MORE THAN ONCE, THE NUMBER OF EDGES THAT TOUCH EVERY VERTEX MUST BE **EVEN**.



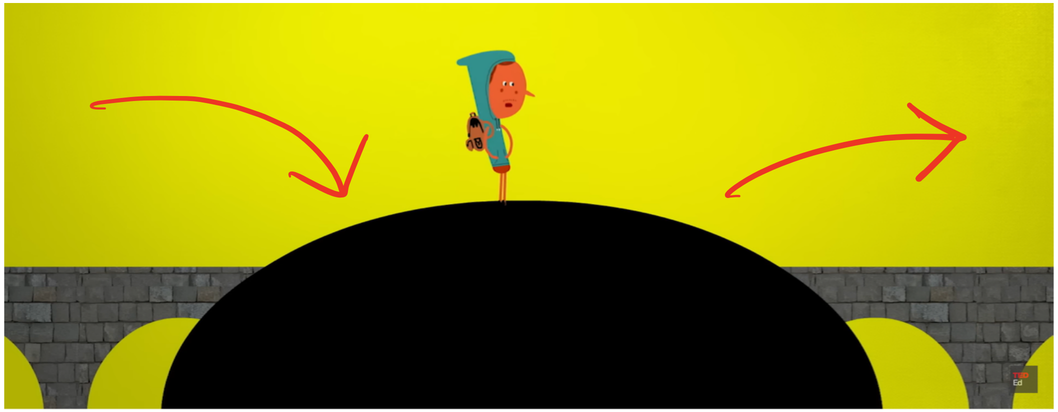


IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

EULERIAN CIRCUIT

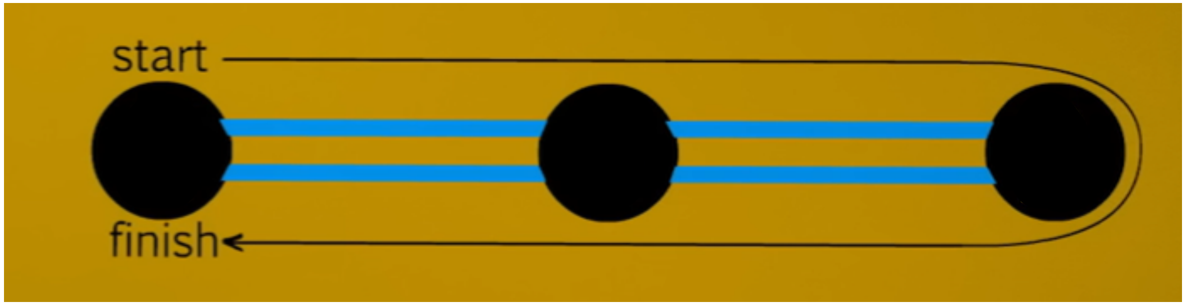


THE COMMON STARTING AND ENDING POINT MAY BE VISITED MORE THAN ONCE; EXCEPT FOR THE VERY FIRST TIME WE LEAVE THE STARTING VERTEX, AND THE LAST TIME WE ARRIVE AT THE VERTEX, EACH SUCH VISIT USES EXACTLY 2 EDGES.

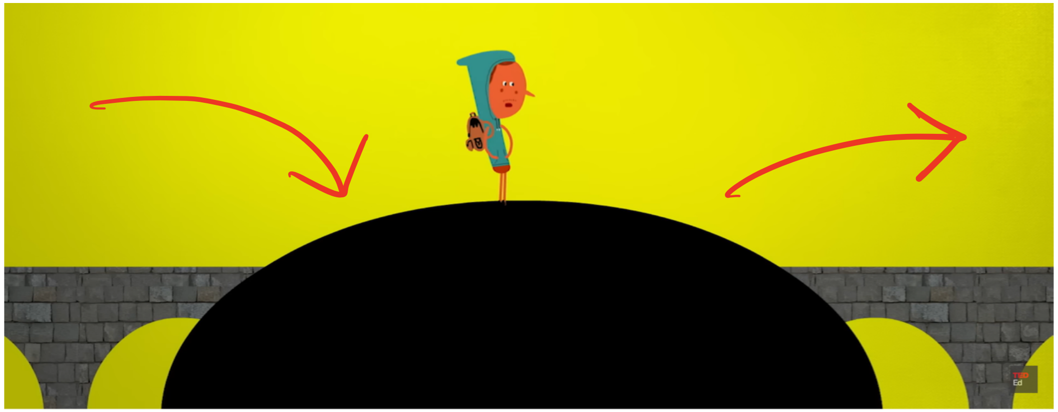


IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

EULERIAN CIRCUIT

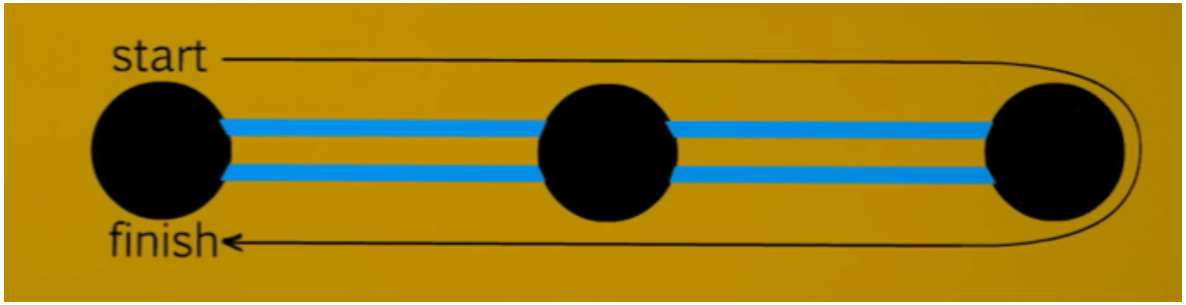


THE STARTING VERTEX MUST BE TOUCHED BY AN EVEN NUMBER OF EDGES.

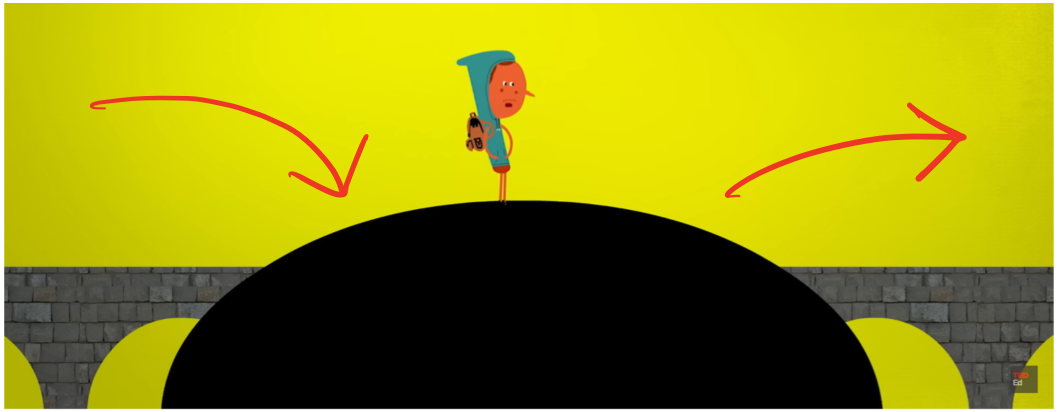


IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

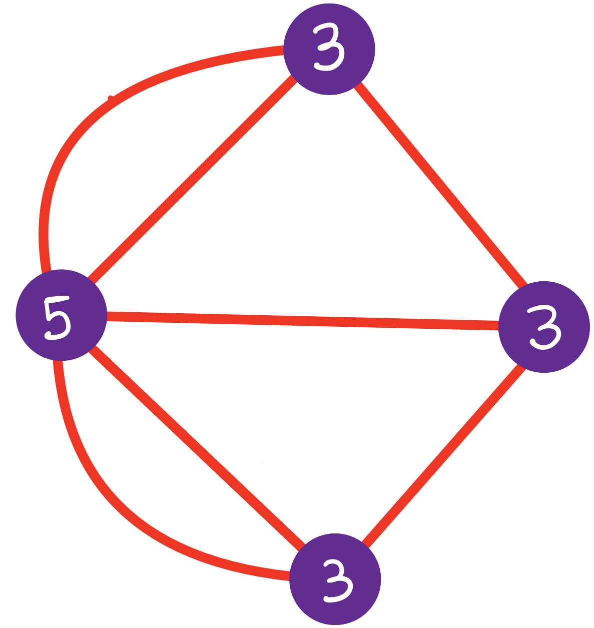
EULERIAN CIRCUIT



EVERY VERTEX MUST BE TOUCHED BY AN EVEN NUMBER OF EDGES.



THE KÖNIGSBERG BRIDGE PROBLEM

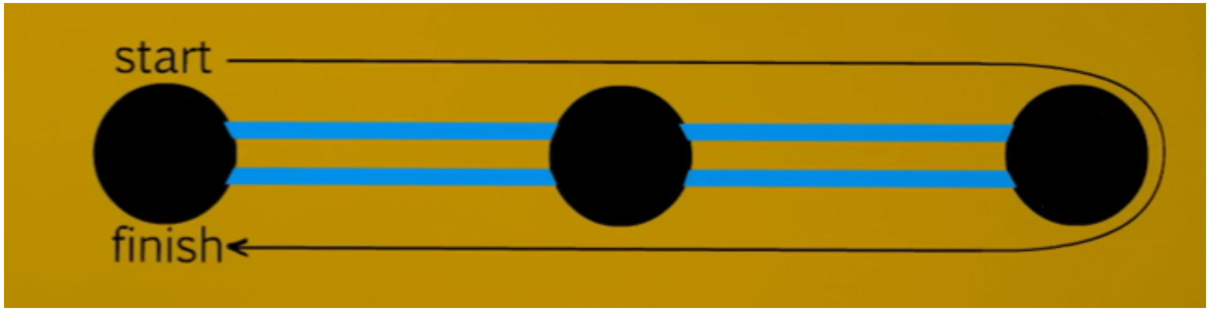


ALL VERTICES HAVE ODD DEGREE

THE DESIRED WALK DOES NOT EXIST!!!

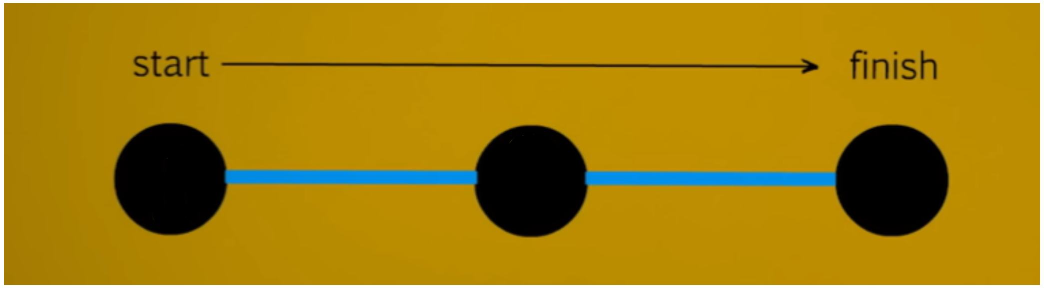
THEOREM

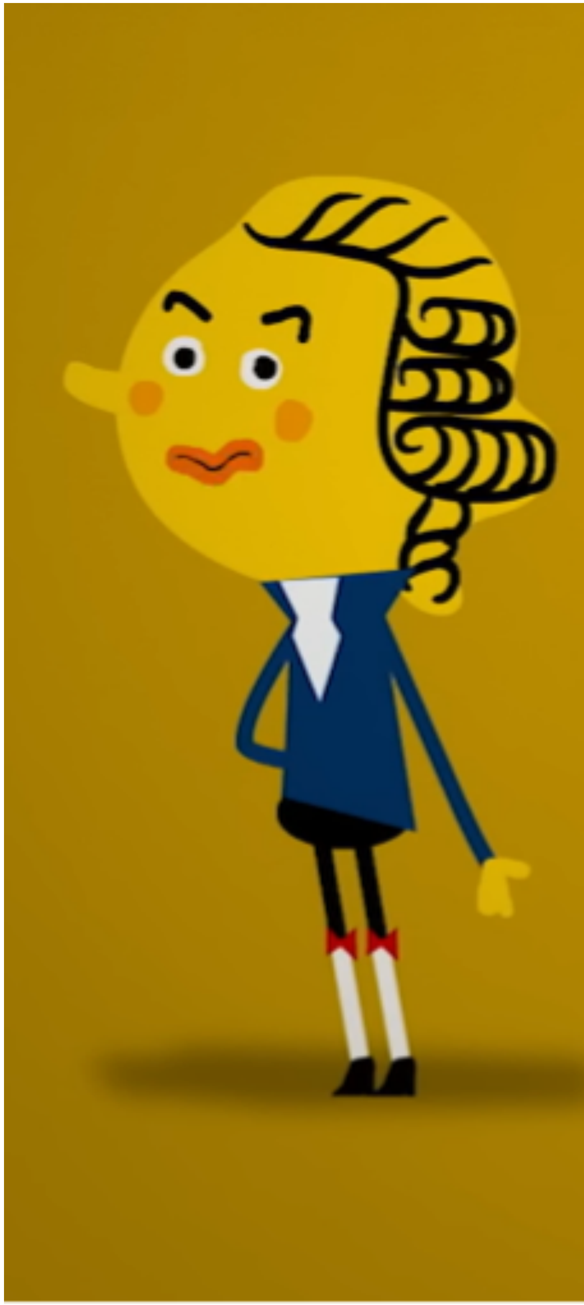
IF G IS A CONNECTED GRAPH, THEN G CONTAINS AN EULERIAN CIRCUIT IF AND ONLY IF EVERY VERTEX HAS EVEN DEGREE.



IMAGINE WE HAVE A WALK IN A CONNECTED GRAPH WHERE EVERY VERTEX IS USED EXACTLY ONCE.

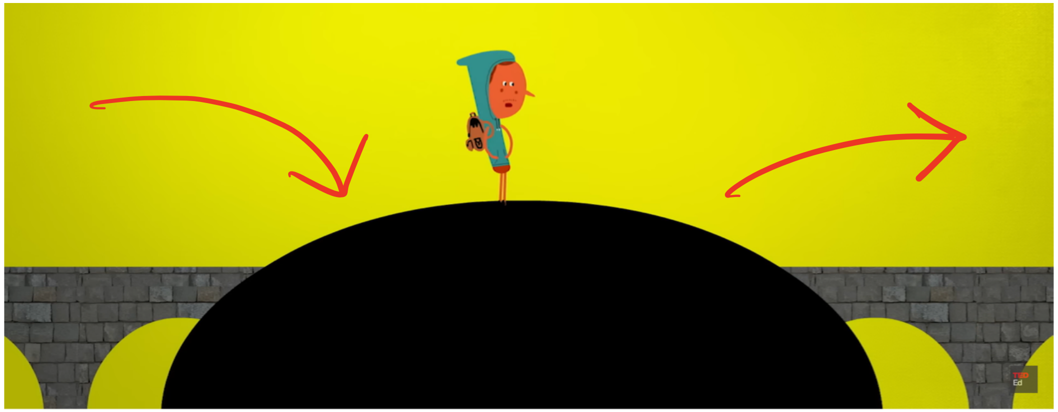
EULERIAN WALK



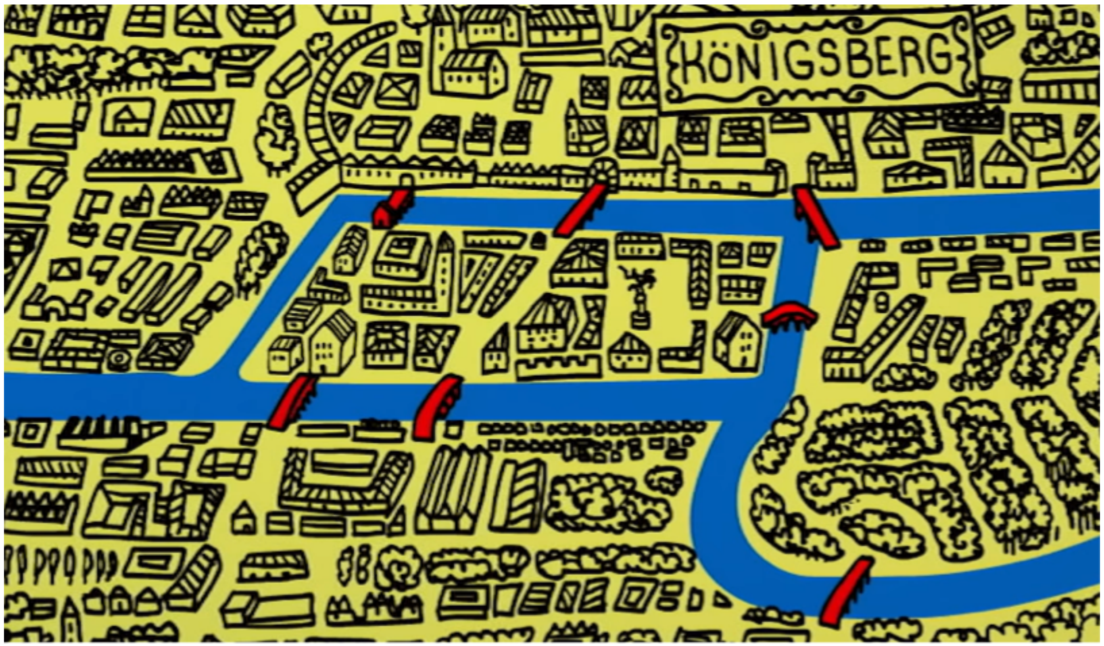


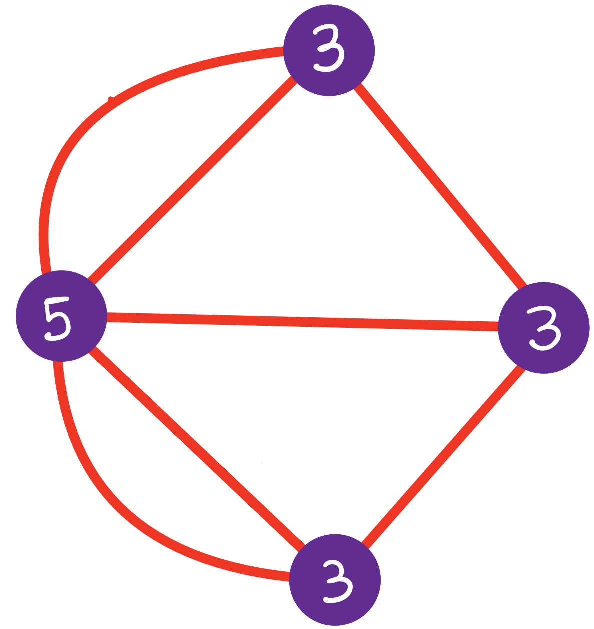
THEOREM

A CONNECTED GRAPH G HAS AN EULERIAN WALK IF AND ONLY IF EXACTLY TWO VERTICES HAVE ODD DEGREE.



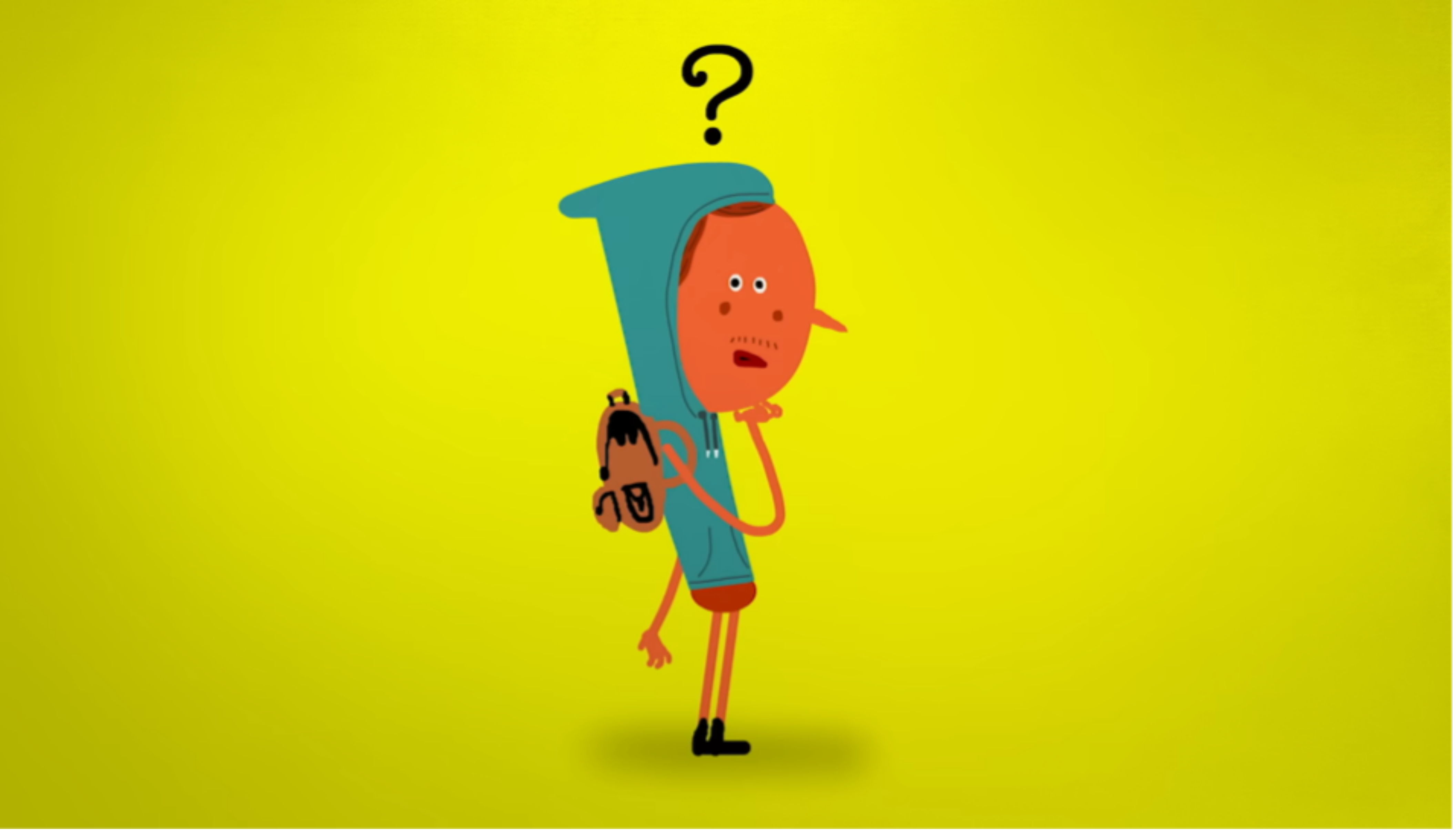
THE KÖNIGSBERG BRIDGE PROBLEM





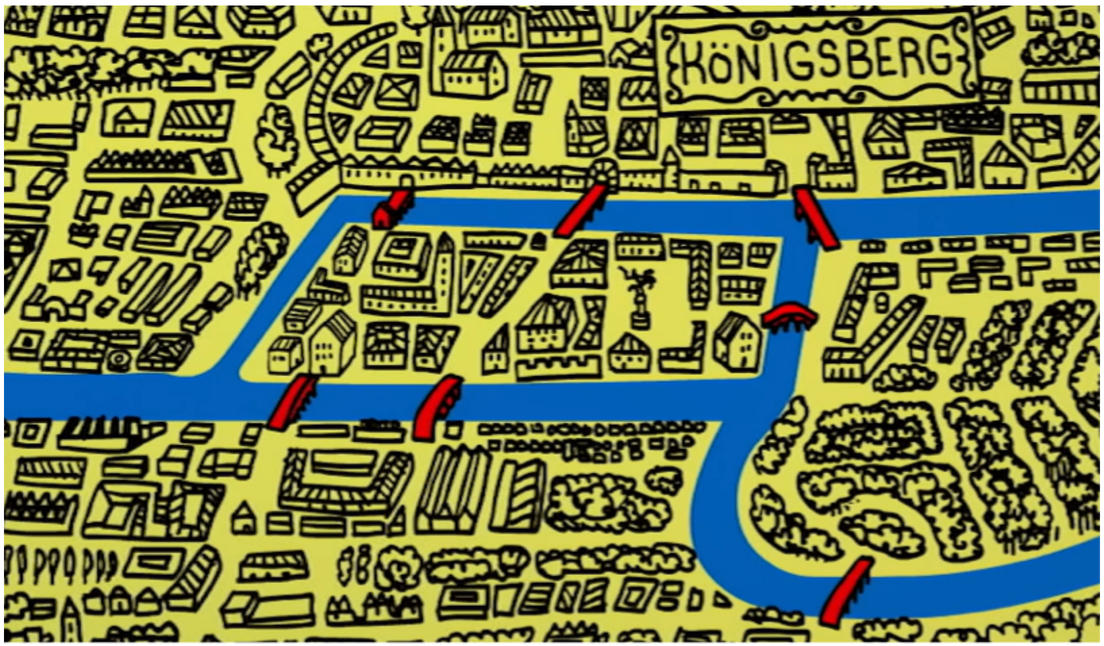
4 VERTICES HAVE ODD DEGREE

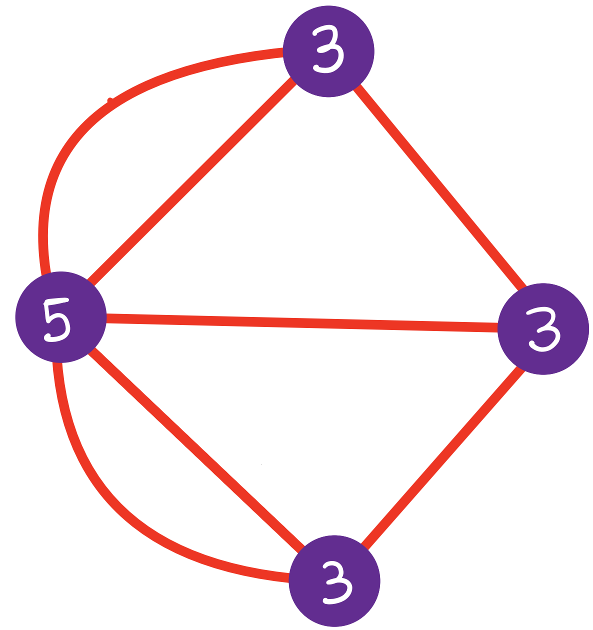
THE DESIRED WALK DOES NOT EXIST!!!



HOW MIGHT YOU CREATE A DESIRED WALK IN KÖNIGSBERG?

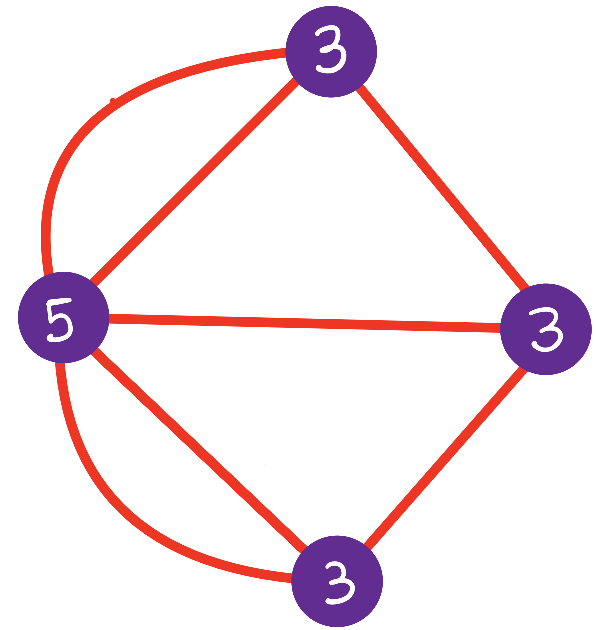
THE KÖNIGSBERG BRIDGE PROBLEM





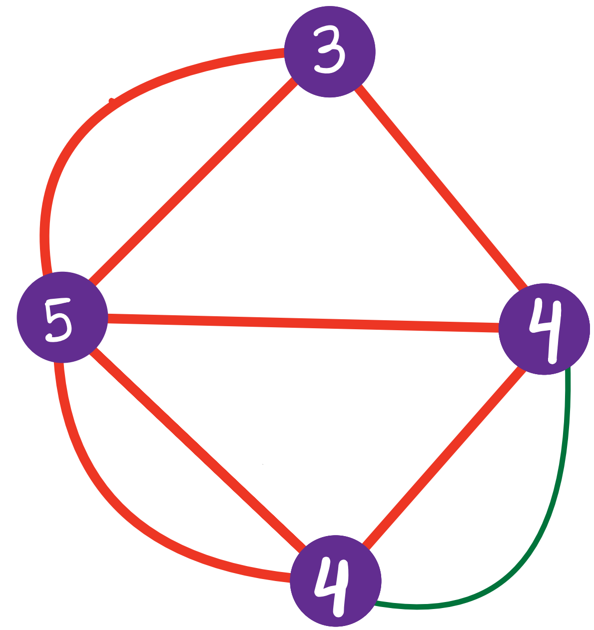
ANY TWO VERTICES HAVE ODD DEGREE.

THE KÖNIGSBERG BRIDGE PROBLEM



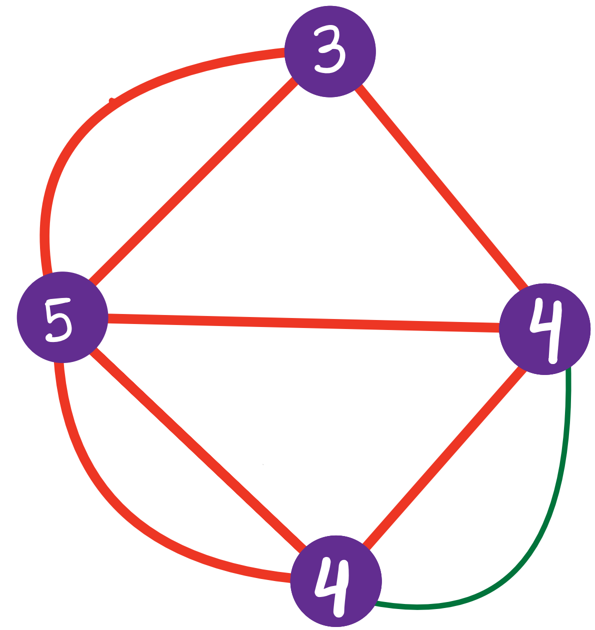
ADD / DELETE AN EDGE BETWEEN ANY TWO VERTICES

THE KÖNIGSBERG BRIDGE PROBLEM

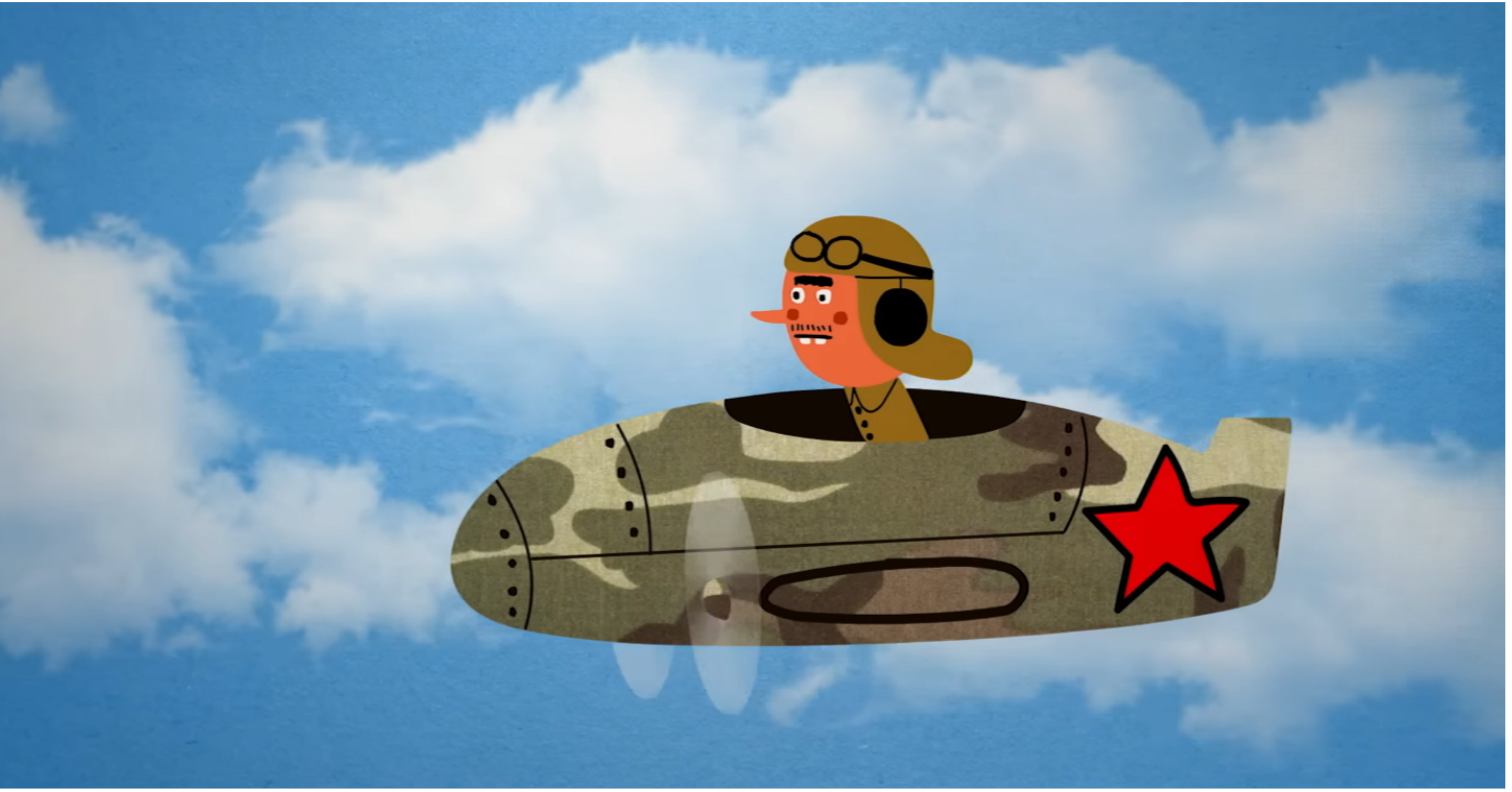


THIS WILL GIVE US A GRAPH WITH 4 VERTICES WHERE EXACTLY TWO ODD VERTICES (AND TWO EVEN VERTICES)

THE KÖNIGSBERG BRIDGE PROBLEM



SUCH A GRAPH WILL HAVE AN EULERIAN WALK...



IT TURNS OUT THAT HISTORY CREATED A EULERIAN WALK OF ITS OWN

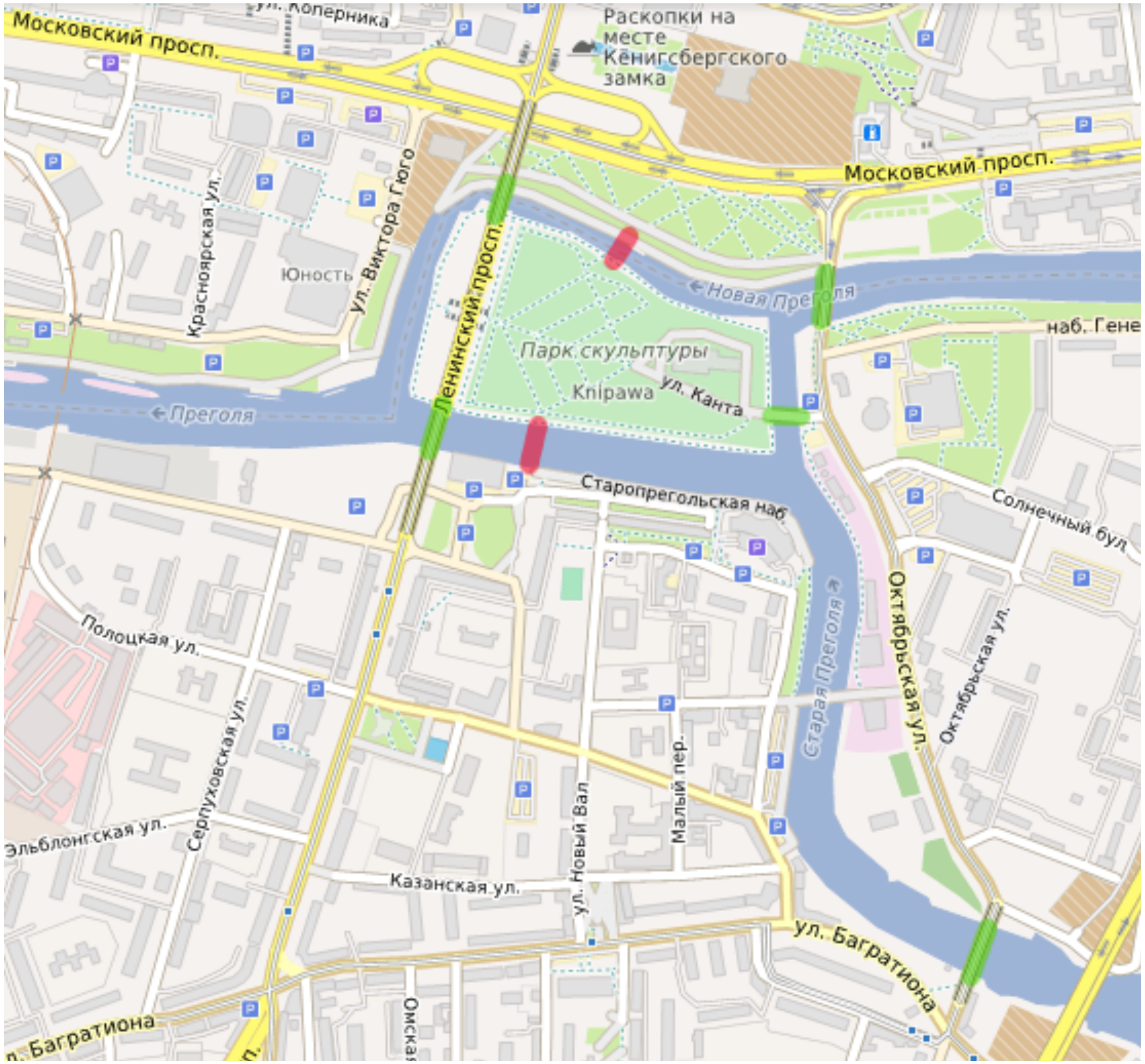
During World War II, the Soviet Air Force destroyed two of the city's bridges

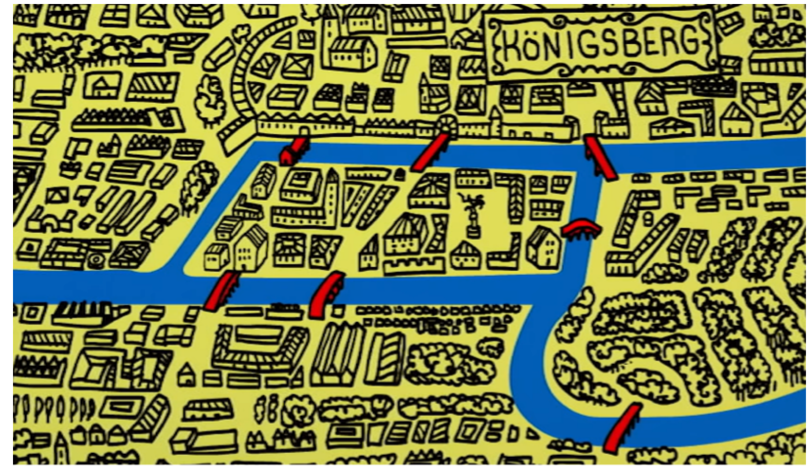
MAKING A EULERIAN WALK EASILY POSSIBLE.

BUT TO BE FAIR, THAT PROBABLY
WAS NOT THEIR INTENTION

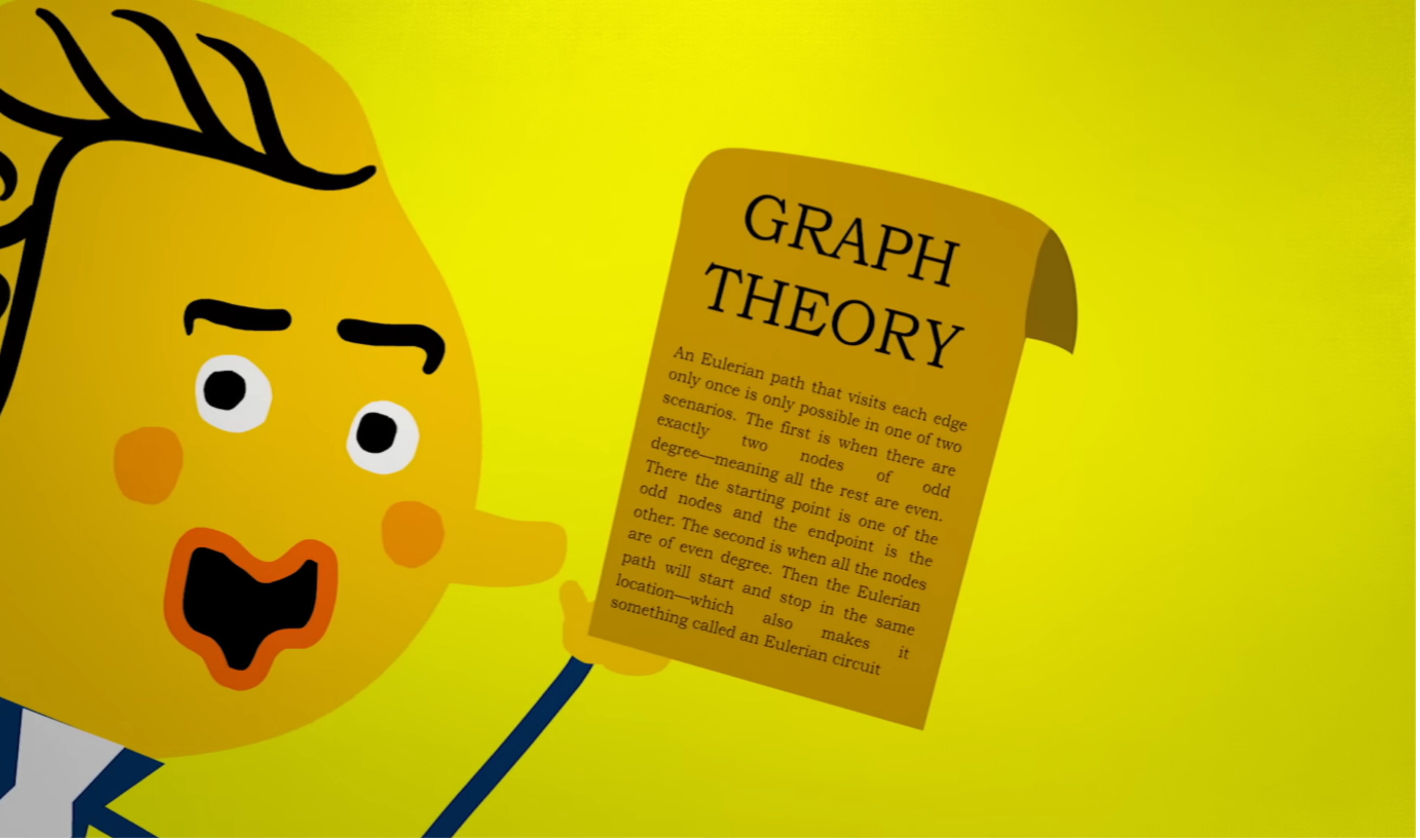
KÖNIGSBERG WAS LATER REBUILT AS THE RUSSIAN CITY OF KALININGRAD.

Present State of the Bridges



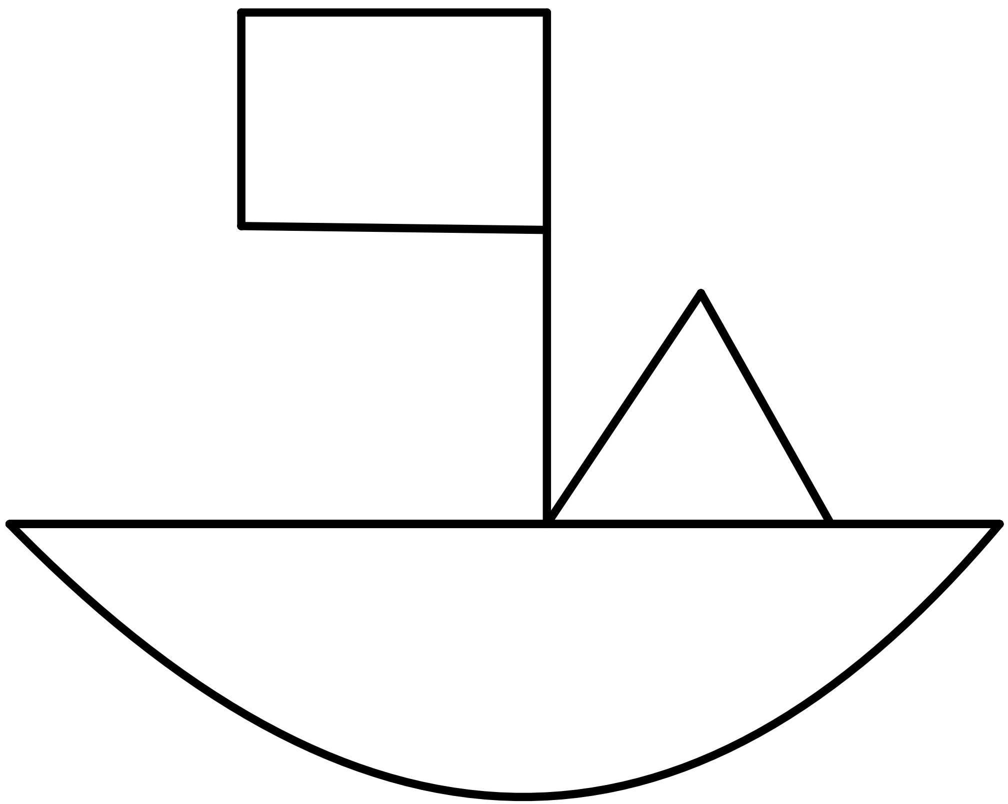


SO WHILE KÖNIGSBERG AND HER SEVEN BRIDGES MAY NOT BE AROUND ANYMORE ...

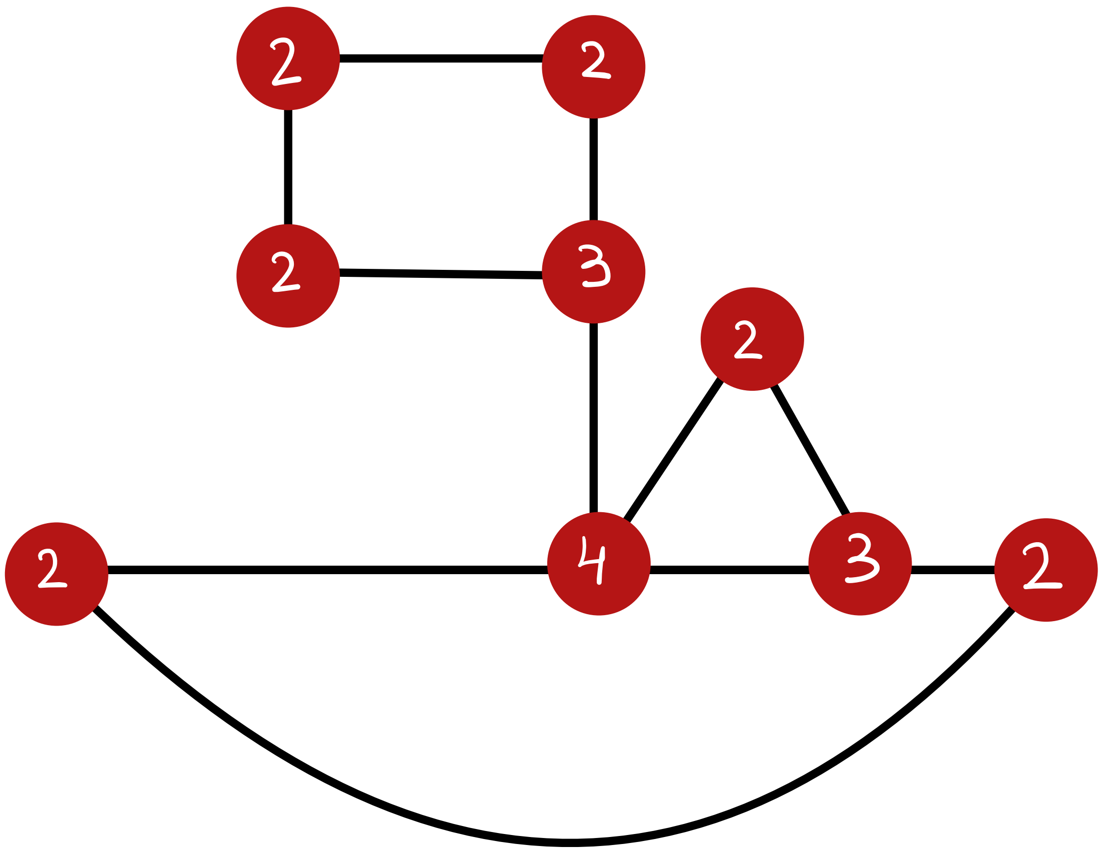


THEY WILL BE FAMOUS IN HISTORY BECAUSE OF A SIMPLE PUZZLE THAT MADE A NEW KIND OF MATH!

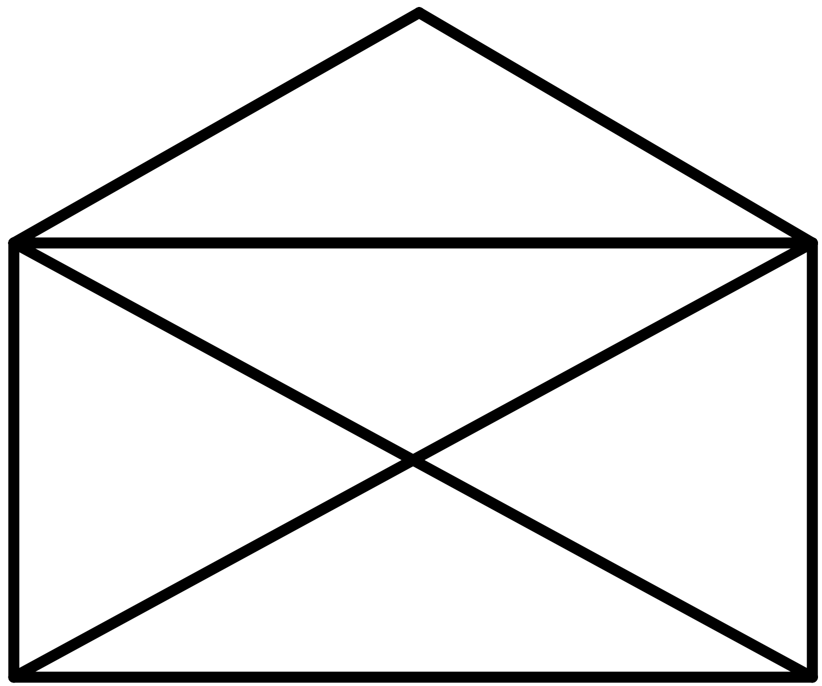
Without Lifting the Pencil



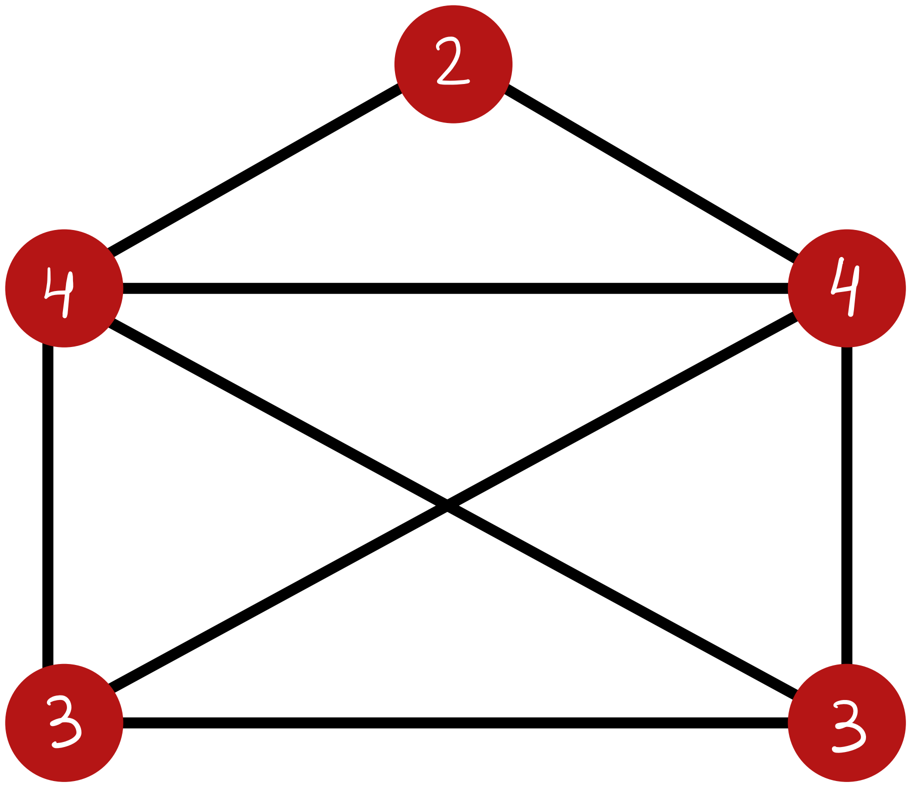
Without Lifting the Pencil



Without Lifting the Pencil



Without Lifting the Pencil



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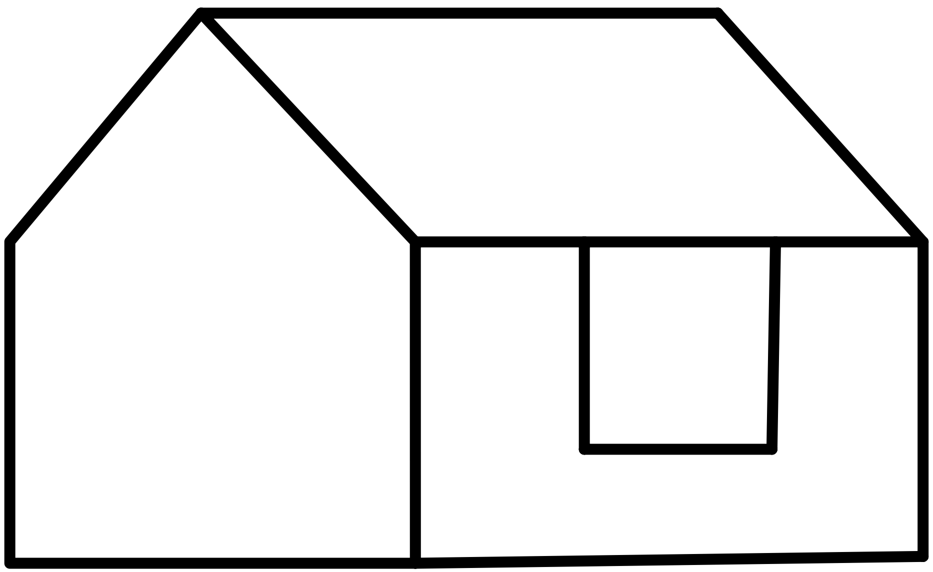

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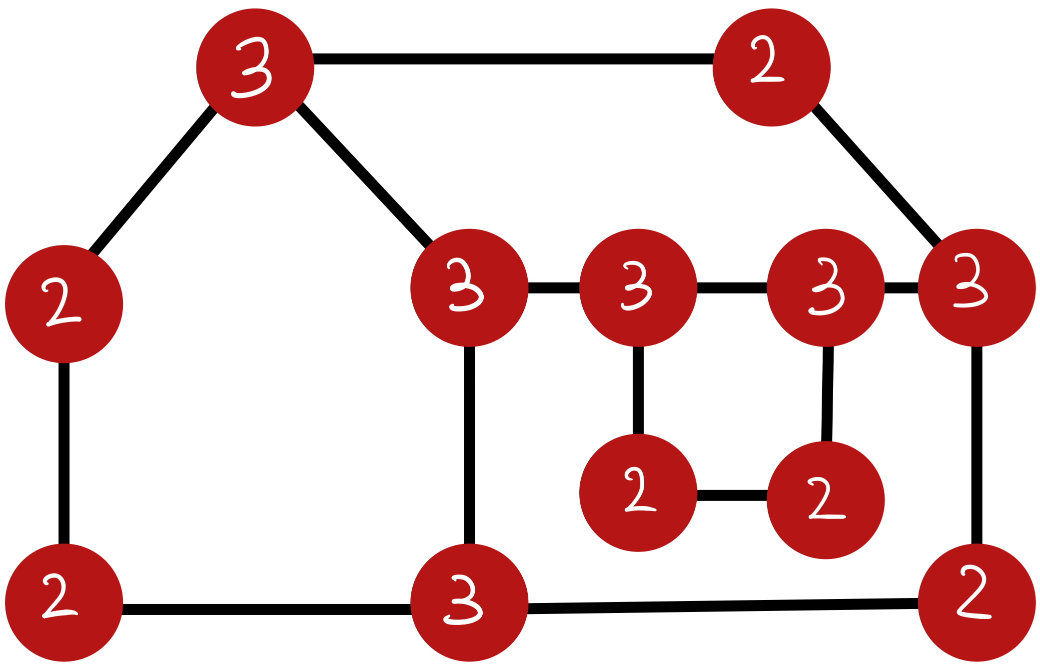

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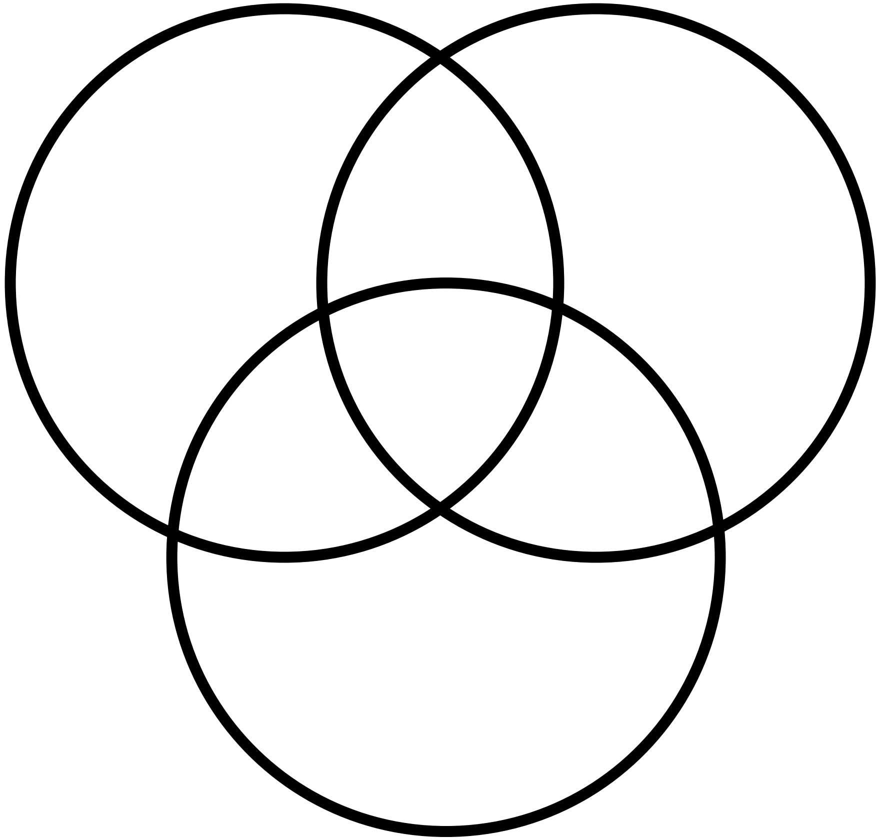

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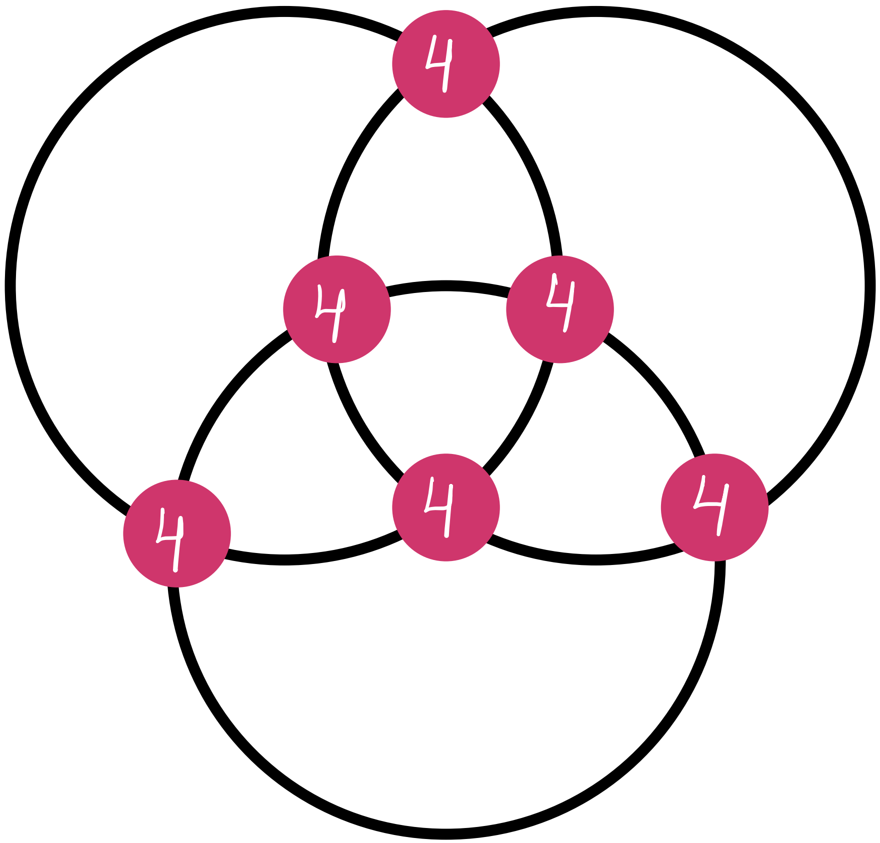

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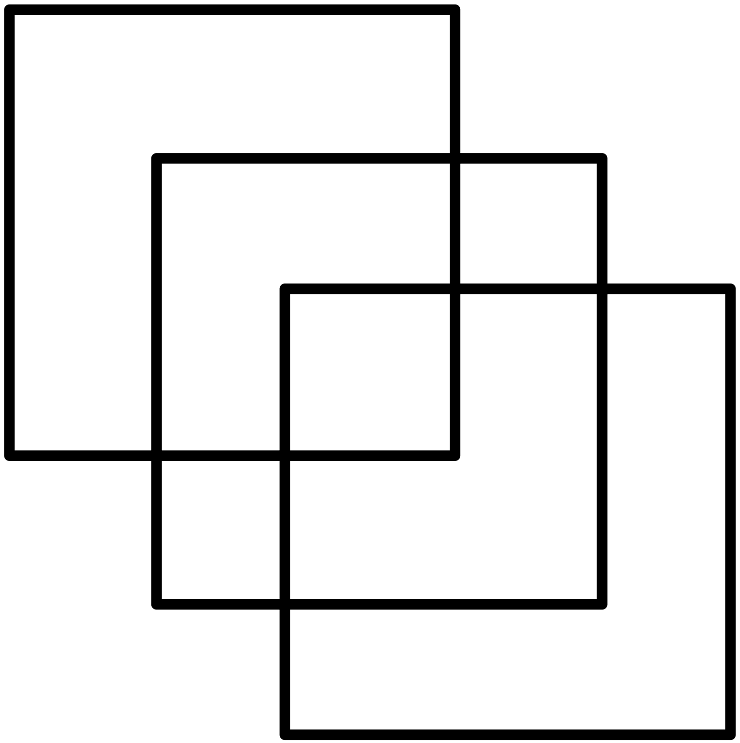
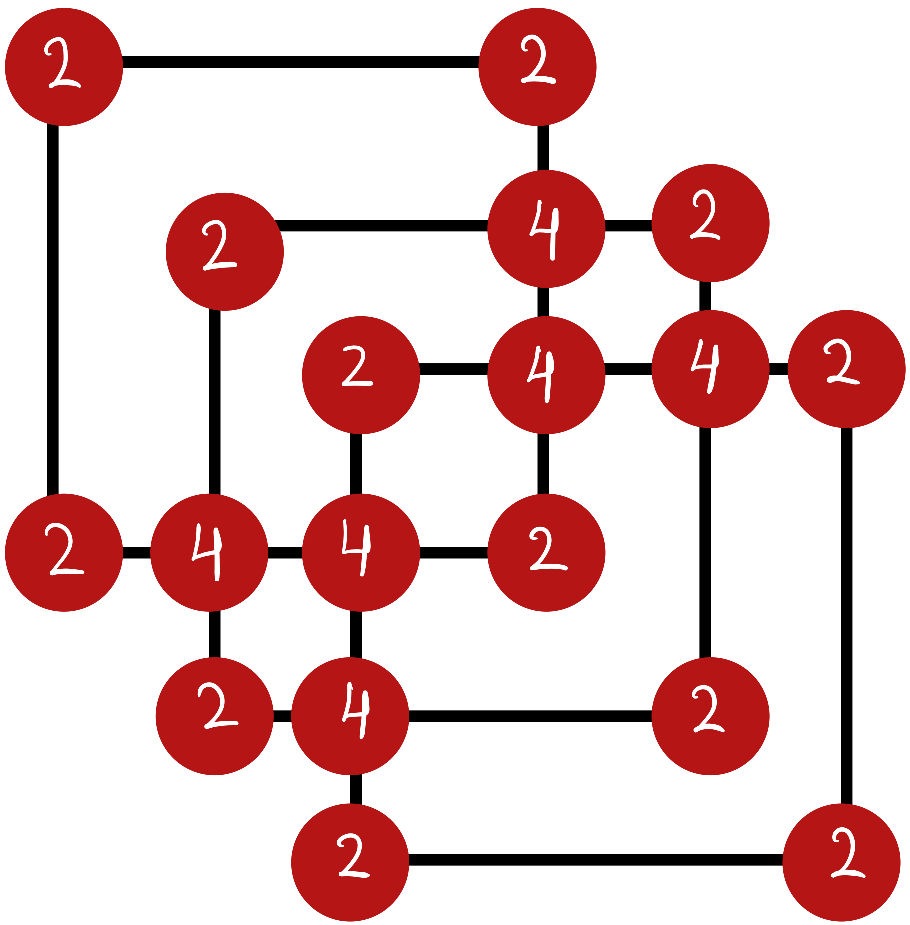


Without Lifting the Pencil



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Thank You!